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Solved Questions.

4487. (B. Williamson, M.A.) — Suppose F, F_1, F_2 to be the three foci of a Cartesian oval, F_2 being the exterior focus, and let $FF_1 = c_2$, $FF_2 = c_1$, $F_1F_2 = c$; then, if $mr + lr_1 = nc_2$ be the equation of the inner oval referred to F and F_1, show that its equation referred to F and F_2 is $nr + lr_2 = mc_1$; and referred to F_1 and F_2 is $mr_2 - nr_1 = lc$. Also prove that the corresponding equations of the outer oval are obtained by changing l into $-l$ in the preceding 116

4759. (Professor Hudson, M.A.)—A right circular cylinder is made of elastic material attached to rigid fixed plane ends. It is distended by fluid pressure. Supposing that the tensions in the circular sections are regulated by Hooke's law, obtain equations sufficient to determine completely the shape it will assume. If the pressure p be constant, prove that the meridian curve is

$$x + A = \int \frac{\frac{1}{2}py^2 + B}{\left\{(\lambda y^2/2a - \lambda y + C)^2 - (\frac{1}{2}py^2 + B)^2\right\}^{\frac{1}{2}}}\,dy \quad \ldots\ldots 42$$

5021. (Professor Hudson, M.A.)—In a fluid at rest under the action of forces of which $X\delta m$, $Y\delta m$, $Z\delta m$ are the components parallel to the axes acting upon a small mass δm, the locus of points about which the greatest variation of pressure is a maximum or a minimum is determined by

$$\frac{X\frac{d\rho X}{dx} + Y\frac{d\rho X}{dy} + Z\frac{d\rho X}{dz}}{X} = \frac{X\frac{d\rho Y}{dx} + Y\frac{d\rho Y}{dy} + Z\frac{d\rho Y}{dz}}{Y} = \frac{X\frac{d\rho Z}{dx} + Y\frac{d\rho Z}{dy} + Z\frac{d\rho Z}{dz}}{Z},$$

ρ being the density at x, y, z 65

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5266. (J. J. Walker, M.A.)—If the line which is the locus of the equation $lx+my+nz=0$ meet the sides BC, CA, AB of the triangle of reference in the points A′, B′, C′, and the bisectors of the angles A, B, C in the points D, E, F respectively, prove that

$$m : n = \text{C′A′/C′D} : \text{B′A′/B′D}, \quad n : l = \text{A′B′/A′E} : \text{C′B′/C′E},$$
$$l : m = \text{B′C′/B′F} : \text{A′C′/A′F} \ldots\ldots\ldots\ldots 64$$

6366. (G. J. Griffiths, M.A.) — Prove that the sum of the squares of the reciprocals of all numbers which are not divisible by the square of any prime is $15\pi^{-2}$ 27

6481. (C. W. Merrifield, F.R.S.) — Obtain as a determinant, in a symmetrical form, the Cartesian equation of a right line passing among any n given points $(x_1, y_1), \ldots, (x_n, y_n)$, so that the sum of the squares of their distances from it shall be a minimum 58

8474. (Professor Gilliland, M.A.) — A closed conical vessel whose height is twice the diameter of the base is just filled with liquid and suspended from a point in the rim of its base. Prove that, if the weight of the vessel be neglected, the vertical component of the pressure on its curved surface will be $\frac{11}{8}$ of the weight of the liquid 47

8625. (A. Gordon.)—Show that, if six points are taken on the sides of a tetrahedron, then the four spheres similar to that one through A and the three points on AB, AC, AD will meet in a point 28, 41

9081. (Professor Wolstenholme, M.A., Sc.D.) — At each point P of a given equiangular spiral is drawn a curve similar to the curve $r^n = a^n \sin n\theta$, having four-point contact with the spiral; S is the pole of the spiral, S′ that of the osculating curve; prove that $SS' = n \cdot SP$, and that the angle PSS′ is double the angle between S′P and the tangent at P 99

9143. (A. E. Thomas.)—A circle described through the origin, and touching the curve $(x^2+y^2-2ax)^2(x^2+y^2-5ax) = 4a^3y^3$ at a point Q, cuts the parabola $y^2 = 4ax$ in three points, the normals at which to the parabola meet in a point W. Find the locus of W as Q varies 49

9448. (B. F. Finkel.)—A dog, on the top of a conical hill whose base is 200 feet in radius and height 100 feet, is in pursuit of a fox at the base which keeps running around the hill at the base; find how far the dog must run to catch the fox, if the dog runs 6 feet while the fox runs 5 feet and the dog runs directly towards the fox at all times 30

9515. (R. Holmes, B.A.)—If P_n is Legendre's coefficient of order n, and J_0 is the Bessel's function of order zero, show that

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$$= e^{-\mu r} J_0\{r(1-\mu^2)^{\frac{1}{2}}\} \ldots\ldots 46$$

9656. (J. P. Johnston, B.A.) — The four lines joining the points where the Hessian of a cubic surface touches three coplanar lines in the surface intersect the Hessian again in four collinear points 83

9699. (Professor Beyens.)—Si le rapport à l'hemisphère du segment *d'une* base de la sphère est m/n, le rapport au rayon de l'hauteur du segment *à deux* bases qui resultera est égal à $2\sin\frac{1}{2}[\sin^{-1}(n-m)/n]$ ... 59

9774. (E. Borel.) — On considère un pentagone quelconque. On trace les cercles qui passent chacun par les extrémités d'un côté et par le point de rencontre des deux côtés adjacents. Démontrer que les cercles contigus se coupent deux à deux en cinq points (autres que les sommets du pentagone) qui se trouvent sur une même circonférence 58

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10143. (Professor Déprez.)—Sur les côtés BC, CA, AB du triangle sphérique ABC on prend, dans le même sens, trois arcs égaux, BA′ = CB′ = AC′. Démontrer que le point de rencontre des médianes du triangle A′B′C′ décrit un grand cercle 115

10542. (W. J. Greenstreet, M.A.)—Three men A., B., C. play a match. They draw lots to see which pair plays first, and the winner of the first game plays the third. A., winning the second game, wins the match. Show the chance of A. and B. having played the first game is $\frac{5}{19}$, when the skills of A., B., C. are as 4 : 3 : 2. Find the chance A. and B. did not play together 28

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14824. (R. Tucker, M.A.)—ABC is a triangle. AD is the perpendicular from A on BC. A circle, with A as centre and AD as radius, cuts AB, AC in points which are joined. The same is done for the other angles. Prove that these three lines (1) intersect in p, q, r, which lie on the bisectors of A, B, C; (2) they form a triangle similar to that formed by the points of contact of the in-circle with the sides of ABC; (3) this in-centre is the orthocentre of pqr; (4) perpendiculars from p, q, r on BC, CA, AB meet in the circum-centre of pqr 93

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14977. (E. Lasker, Ph.D.)—Prove that, if A, B, C, D are four fixed points on a sphere, P a variable one, τ its tangent plane; and if the angle formed by the intersections of PAC, PBC on τ is γ, and the other one formed in the same manner by PAD, PBD is δ, then $\gamma - \delta$ is (in regard to P) a constant magnitude 71

15077. (Professor Cochez.)—On donne une circonférence O et un point A, par lequel on mène une sécante variable coupant la courbe en B et C. Lieu de l'orthocentre du triangle BOC 32

15083. (John Prescott, B.A.)—Given two lines AB and CD in the same plane, but not parallel, find a point P such that the triangles formed by joining P to the extremities of AB and CD shall be similar, the two given lines subtending equal angles at P 111

15095. (Professor Neuberg.)—ABC étant un triangle donné, trouver un point M tel que si A′, B′, C′ sont les points de rencontre de BC, CA, AB avec les droites AM, BM, CM, M soit le centre de gravité des points A′, B′, C′ pour les masses données α, β, γ 116

15120. (Professor Cochez.)—On considère la parabole $y^2 = 2px$. Lieu des points P tels que, menant les tangentes PM, PM′ rencontrant les axes en A, D, B, C, le quadrilatère ABCD soit circonscriptible ... 77

15121. (R. Knowles.)—Two tangents are drawn to meet a rectangular hyperbola centre C; a third tangent meets these in M, N; O is the mid-point of MN; F, G are the projections of O on the asymptotes; O′ is the mid-point of FG. Prove that (1) the points O, O′, C are collinear; (2) if H is the projection of C on FG, the angle HCO is bisected by the axes 50

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15133. (John Prescott, B.A.) — Through P, a point within the angle BAC, draw a line MPN, meeting AB and AC in M and N respectively, such that the sum of AM and AN shall be equal to a given length. Also draw the line so that the sum of AM and AN may be a minimum 34

15147. (Lt.-Col. Allan Cunningham, R.E.)—Let $\Sigma(p, a)$ = sum of numbers $\neq X$ containing p^a, but not $p^{a+\lambda}$, where

$$X = (p^m + q^m)^n - (p^m + q^m).$$

Prove that, when $a+\lambda$, $b+\lambda$ are both $\neq m$,

$$\Sigma(pq, b)/\Sigma(pq, a) = \Sigma(p, b) \cdot \Sigma(q, b)/\Sigma(p, a) \cdot \Sigma(q, a) \ldots\ldots 97$$

15148. (R. W. D. Christie.)—Prove

$(1^{\phi(a)} - 1)^n = 0 \bmod a_1 a_2 a_3 \ldots a_n$, where $\phi(a_1) = \phi(a_2) = \ldots = \phi(a_n)$ 29

15163. (Rev. T. Roach, M.A.)—If A, B, C be the angles of a triangle, show that

$$\begin{vmatrix} \sin A & \cos A & \cos(C-A) \\ \sin B & \cos B & \cos(B-C) \\ \sin C & \cos C & 1 \end{vmatrix} = 0 \ldots\ldots 46$$

15171. (A. M. Nesbitt, M.A.)—Discuss the curves—

(1) $(y-x^2)^2 = xy^4$, (2) $(y-x^2)^2 = x(x^4-y^4)$ 100

15177. (R. Tucker, M.A.) — In the "ambiguous" case AC = AC', AL, AL' are the respective symmedians (through A) of the triangles ABC, ABC', cutting BC in L, L' respectively. Find (i.) LL', (ii.) the value of BC when AC' is the symmedian of ABC 49

15195. (R. Tucker, M.A.)—Prove

$$\begin{vmatrix} \cos B \cos C & -p \cos C & p \cos B \\ p' \cos C & \cos C \cos A & -p' \cos A \\ -p'' \cos B & p'' \cos A & \cos A \cos B \end{vmatrix} = \Pi(\cos A)\Pi(\cos B + \cos C - \cos A),$$

where $p = \cos B - \cos C$, $p' = \cos C - \cos A$, $p'' = \cos A - \cos B$ ... 119

15218. (James Blaikie, M.A.) — If x be the edge of an icosahedron and y the radius of the circum-sphere, show that $5x^4 - 20x^2y^2 + 16y^4 = 0$ 102

15222. (I. Arnold.) — There are two inclined planes each 100 feet long, viz., AC and BC, and each inclined at an angle of 15° to the horizon. A small smooth weight slides freely down AC and then ascends BC; after descending BC it reascends AC; and so on. It is required to find the number of times it will ascend the plane AC before losing half the velocity acquired in the first descent 77

15238. (Professor Sanjána, M.A.)—Given of a triangle the base and the vertical angle, construct the triangle so that the ratio may be fixed of the distance of the mid-point of the base from the foot of the perpendicular drawn from the vertex to its distance from the foot of the bisector of the vertical angle 61

15241. (Professor Sanjána, M.A.)—In a plane triangle ABC, (1) if the sides a, b include an angle $\cos^{-1}\sqrt{(ab)}/[a-\sqrt{(ab)}+b]$, prove that $\tan A/\tan B = -(a^3/b^3)^{\frac{1}{3}}$, so that one of the remaining angles is obtuse; (2) if tan C, tan A, tan B be in geometric progression, the common ratio is b^2/c^2, and A is greater than $\frac{1}{3}\pi$; (3) if $\tan\frac{1}{2}C$, $\tan\frac{1}{2}A$, $\tan\frac{1}{2}B$ be in geometric progression, the common ratio is b/c, and A is less than $\frac{1}{3}\pi$ 26

15250. (Lt.-Col. Allan Cunningham, R.E.)—State conditions under which one, or both, of $N_1 = (a^n-1)^3-1$, $N_2 = (a^n+1)^3+1$ are completely resolvable into *prime* factors, when $a^n > 3000$. *Ex.*—Show that both are possible when $a^n = 2^{70}, 3^{15}, 5^{10}, 10^7$ 47

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15260. (A. M. Nesbitt, M.A.)— Given a point P within an angle BAC, draw two equal circles through P, one of which shall touch AB and the other AC 73

15269. (Lt.-Col. Allan Cunningham, R.E.)—(Extension of Question 15147.)—If $X = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$. $X' = X'\Pi(p_r^{m_r})$ and $\Sigma(p_r, m)$, $\Sigma(p_r, n)$ denote the sum of the numbers $\not\vdots$ X containing p_r^m, p_r^n, but not $p_r^{m+\lambda}$, $p_r^{n+\lambda}$, respectively, as factors, show that, when $m+\lambda$, $n+\lambda$ are both $\not\vdots$ the least m_r, then

$$\Sigma(\Pi p_r, m) : \Sigma(\Pi p_r, n) = \Pi\Sigma(p_r, m) : \Pi\Sigma(p_r, n) \quad \dots\dots 88$$

15275. (R. Tucker, M.A.)—ABC is a triangle, and lines are drawn through the angles, making with the sides angles $= \theta$ and triangles abc, $a'b'c'$. Prove that the circles abc, $a'b'c'$ have for their equations

$$\Delta\,\Sigma a\beta\gamma = R \sin\theta\, \Sigma a\alpha\, \Sigma c \sin(B-\theta)\,\alpha$$

and

$$\Delta\,\Sigma a\beta\gamma = R \sin\theta\, \Sigma a\alpha\, \Sigma b \sin(C-\theta)\,\alpha.$$

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15277. (Professor Cochez.)—Étudier les courbes

(1) $y^2 = x^4/(1-x)$, (2) $y = 3x+5+(x-2)/(x^2-8x+15)$ ... 79

15279. (Professor Sanjána, M.A.)—Two distinct sections of a cone are made by planes perpendicular to an axial plane, so as to have the same eccentricity and a common vertex; prove that their *latera recta* vary as their major axes 29

15282. (Hon. G. R. Dick, M.A.)—Prove that, if a is any real quantity, positive or negative,

$$\int_0^\infty \frac{\sin ax}{1+x^2}\,dx = \tfrac{1}{2}e^{-a}\,\mathrm{li}\,e^a - \tfrac{1}{2}e^a\,\mathrm{li}\,e^{-a},$$

where $\mathrm{li}\,e^a$ is the logarithmic integral and may be defined by

$$\mathrm{li}\,e^a = \mathrm{Ei}(a) = \int_0^1 \frac{e^{az}}{z}\,dz - \int_0^\infty \frac{e^{-z\sqrt{a^2}}}{z}\,dz,$$

for all real values of a, positive and negative 89

15284. (J. J. Barniville, B.A.) — Having $u_n = u_{n-2} + u_{n-3}$, prove that

$$\frac{1}{2.3.4.9} - \frac{1}{3.4.5.12} + \frac{1}{4.5.7.16} - \frac{1}{5.7.9.21} + \dots = \frac{1}{280},$$

$$\frac{2.5.11}{2.4.7.9} + \frac{3.7.14}{2.5.9.12} + \frac{4.9.19}{3.7.12.16} + \frac{5.12.25}{4.9.16.21} + \dots = \frac{25}{21} \quad \dots\dots 25$$

15285. (Lt.-Col. Allan Cunningham, R.E.) — If y, Y are *proper* roots of

$$(y^n-1) \div (y-1) \equiv 0 \pmod p,$$

and y', Y' are *proper* roots of

$$(y'^n+1) \div (y'+1) \equiv 0 \pmod p,$$

and, if y, y' are $< \frac{1}{2}p$, and Y, Y' are $> \frac{1}{2}p$, but $< p$, show that (when p, q, r are unequal *odd* primes) then

i. $\Sigma y - \Sigma y' = \Sigma Y - \Sigma Y' \equiv -1$, if $n = q$;

and ii. $\equiv 0$, if $n = q^2$;

and iii. $\equiv +1$, if $n = qr$;

the congruences being all for modulus p, and the summations including all incongruous roots within the limits stated 80

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15286. (R. W. D. Christie.) — Extend Euler's theorem of two squares to the cubes: *e.g.*, $(1^3+19^3)(17^3+37^3) = 683^3+397^3$ 76, 99

15288. (Rev. Prebendary W. A. Whitworth, M.A.) — Show that there are *only* five triangles whose sides are integers and whose area is represented by the same integer as the sum of the sides 54, 62

15297. (C. M. Ross.)—P is a point in the diameter of the semicircle ACDB, PC is drawn at right angles to AB, and PEB is a semicircle on the same side of AB. If QR be a diameter of the circle which touches PC in Q and the semicircles in D and E, show that the rectangle AB . QR is equal to the square on CP 33

15300. (G. H. Hardy, M.A.)—If $0<\alpha<\frac{1}{2}\pi$, $0<\beta<\frac{1}{2}\pi$, and $0<s<1$,

$$\int_0^{\frac{1}{2}\pi}\cos^{-s}(\theta-\alpha)\cos^{s-2}(\theta-\beta)\,d\theta = \operatorname{cosec}(\beta-\alpha)\left\{\left(\frac{\cos\alpha}{\cos\beta}\right)^{1-s}-\left(\frac{\sin\alpha}{\sin\beta}\right)^{1-s}\right\}$$

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15308. (A. M. Nesbitt, M.A.) — The two tangents (other than the y-axis) which can be drawn from the point $(0, b)$ to the curve $y^3+x^3 = ax^2$ have for equation $4ax^2 = 27b^3(x/a+y/b-1)^2$ 118

15315. (Constance I. Marks, B.A.)—A series of circles touches a fixed straight line AB at a fixed point C, and a straight line EF parallel to AB cuts each of the series. Show that C lies on one or other of the bisectors of the angles between every possible pair of tangents to the series at the points of section of the circles and EF 59

15319. (J. L. S. Hatton.)—Solve the equation $x^4+px^2+qx+r = 0$ by comparing the coefficients with those in the circulant

$$\begin{vmatrix} x & a & b & c \\ c & x & a & b \\ b & c & x & a \\ a & b & c & x \end{vmatrix}$$

equated to zero 106

15322. (Professor Neuberg.) —Trouver les cubes parfaits de la forme $7p+1$, p étant un nombre premier 110

15323. (Rev. T. Roach, M.A.)—Prove that the series

$$\frac{1}{1.2.6.7}-\frac{1}{3.4.8.9}+\frac{1}{5.6.10.11}-\ldots = \frac{1}{60}\left(\pi-\frac{149}{60}\right) \quad \ldots \quad 52$$

15325. (Salutation.) — There is a two-figured number (A) which doubled less 1 becomes reversed; if (A) doubled less 10 be taken from thrice a three-figured number (B), the latter is reversed; and if to (B) be added one-tenth of its value less 1 (remainders being rejected), and the whole be deducted from a four-figured number (C) quadrupled, then (C) is reversed. Find (A), (B), (C) 82

15326. (V. Daniel, B.Sc.)—A man leaving his home for a country station averages 120 yards per minute walking and 200 running, but he misses his train by 5 minutes. If he had run where he walked, and *vice versa*, he would have had 10 minutes to wait, having exceeded his former average for the whole distance by $43\frac{3}{4}$ yards per minute. How long was he on the road? 40

15329. (Professor Sanjána, M.A.) — In a nodal cube, N being the node, a variable chord ABC is so drawn that the tangents at A and B meet on the curve at D. Prove that N (ABCD) is an involution of which the nodal tangents are double rays. (In this case the satellite of ABC touches the curve.) 46

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15332. (A. M. Nesbitt, M.A.)—Any tangent to a hyperbola meets the asymptotes in P and P′. Prove that the locus of the centre of the circle circumscribing the triangle CPP′ is a hyperbola 84

15333. (R. Tucker, M.A.) — ABC is a triangle; AD, BE, CF are the perpendiculars; and on them are taken (internally) lengths $Ad\,(=l)$, $Be\,(=m)$, $Cf\,(=n)$. Find the area of the triangle *def*. Also find the areas (1) if $l=m=n$; (2) if $l=ka$, $m=kb$, $n=kc$ 70

15336. (R. Chartres.)—Express

$$\int_0^{\pi}\int_0^{\theta} \sin^n\theta \sin^n\phi \sin(\theta-\phi)\,d\theta\,d\phi$$

as a single integral of $\sin\theta$ between the limits 0 and $\frac{1}{2}\pi$. Could

$$\int_0^{\pi}\int_0^{\theta} \sin^{n+2}\theta \sin^{n+2}\phi \sin^n(\theta-\phi)\,d\theta\,d\phi$$

be similarly expressed? 32

15339. (James Blaikie, M.A.)—If the odd numbers be arranged in their natural order 1, 3, 5, ..., and divided into groups, such that the numbers in the groups taken in order are 1, 1, 4, 3, 9, 6, ..., $(r-1)^2$, $(r-1)\,r/2$, r^2, ..., and if every second group be struck out, beginning with the second, prove that the sum of the r^2 numbers in the r-th retained group will be r^5. This is obviously an extension of the known theorem that, if the odd numbers be divided into groups of 1, 2, 3, ... numbers, then the sum of the r numbers in the r-th group is r^3. Extend the theorem to the case of groups such that the number of numbers in the r-th group shall be r^3 and their sum r^7 43

15342. (Rev. T. Roach, M.A.)—Sum the series

$$\frac{1}{1.2.4.5} - \frac{1}{3.4.6.7} + \frac{1}{5.6.8.9} - \ldots \quad \ldots\ldots 64$$

15343. (A. M. Nesbitt, M.A.)—If

$$x+y+z=0 \quad \text{and} \quad ax^2+by^2+cz^2=0,$$

prove that, if λ^2 be a certain rational function of a, b, c,

$$[(a+\lambda)^n-(a-\lambda)^n]x^n+[(b+\lambda)^n-(b-\lambda)^n]y^n+[(c+\lambda)^n-(c-\lambda)^n]z^n=0$$

..... 27, 82

15346. (Robt. W. D. Christie.)—Establish a connexion between the following pairs of theorems, and, if possible, the general connexion:—

(1)
$$\begin{cases} \cos x+\cos y+\cos z+\cos(x+y+z) \\ \qquad = 4\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(y+z)\cos\tfrac{1}{2}(z+x), \\ x^2+y^2+z^2+(x+y+z)^2 \\ \qquad = (x+y)^2+(y+z)^2+(z+x)^2; \end{cases}$$

(2)
$$\begin{cases} \cos(A+B-C)+\cos(A-B+C)+\cos(B+C-A)+\cos(A+B+C) \\ \qquad = 4\cos A\cos B\cos C, \\ (a+b-c)^2+(a-b+c)^2+(b+c-a)^2+(a+b+c)^2 \\ \qquad = 4(a^2+b^2+c^2); \end{cases}$$

(3)
$$\begin{cases} \cos A+\cos B+\cos(A+C)+\cos(B+C) \\ \qquad = 4\cos\tfrac{1}{2}(A+B+C)\cos\tfrac{1}{2}C\cos\tfrac{1}{2}(B-A), \\ a^2+b^2+(a+c)^2+(b+c)^2 \\ \qquad = (a+b+c)^2+c^2+(b-a)^2 \end{cases}$$

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15347. (R. Tucker, M.A.)—ABC is a triangle; $\angle BAE = \angle CAD$ and BE, CD, perpendicular to AE, AD respectively, meet in K; then locus of K is a hyperbola about the triangle. Prove that the tangent at A is the "S" line through A, that the fourth point of section is the point in which the join of A to the circum-centre meets the circle, and the three circum-hyperbolas pass through the orthocentre 72

15350. (A. M. Nesbitt, M.A.)—On the tangent at P to an ellipse a point T is taken such that TP = TC; TR and TP are equally inclined to the axis, and CR is drawn perpendicular to TR. If CQ be conjugate to CP, prove that the ratio CR : CQ is constant 45, 81

15353. (H. B. Reynolds.)—In a house are two cisterns respectively 3 ft. cube and 4 ft. cube. It is required to replace these by two new cube cisterns, one less than 3 ft. cube, the other more than 4 ft. cube, so that the two new ones may together contain the same as the two original ones, *i.e.*, 91 cubic feet. The answer must be exact in vulgar fractions —not an approximation in decimals 89

15354. (Robt. W. D. Christie.)—Let P be any prime of form $4m+1$, and r any primitive root; then $r^m(r^m+1) = r^m - 1 \pmod P$. Show the utility of the sequence in arriving at the first prime root 78

15358. (C. M. Ross.)—Find the coefficient of $x^m y^n$ in the expansion of $[(1+x)^p(1+y)^p]/(1+x+y)^p$ 75

15361. (G. H. Hardy, M.A.)—The area of the diagonal triangle of the quadrangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) (in areal co-ordinates) is numerically $\dfrac{54\Delta\, D_{234}\, D_{314}\, D_{124}\, D_{123}}{D_{23,14}\, D_{31,24}\, D_{12,34}}$,

where $D_{\lambda\mu\nu} = \begin{vmatrix} x_\lambda & y_\lambda & z_\lambda \\ x_\mu & y_\mu & z_\mu \\ x_\nu & y_\nu & z_\nu \end{vmatrix}$, $D_{\alpha\beta,\gamma\delta} = \begin{vmatrix} 1 & x_\alpha - x_\beta & x_\gamma - x_\delta \\ 1 & y_\alpha - y_\beta & y_\gamma - y_\delta \\ 1 & z_\alpha - z_\beta & z_\gamma - z_\delta \end{vmatrix}$,

and Δ is the area of the triangle of reference 118

15363. ("Unsigned.")—$\Sigma = 0$, $\Sigma' = 0$ are the tangential equations of two conics of equal area Δ; α is the envelope of a chord of constant length drawn in Σ; α' is the envelope of an equal chord drawn in Σ'. Show that the envelope of the common tangents of α and α' is $k\Sigma\Sigma' = \pi^4/\Delta^4$, and the conic $\Sigma = \Sigma'$, where k is a constant, inserted to keep the dimensions correct; also when the areas are not equal the envelope is a curve of the sixth order 113

15364. (R. Tucker, M.A.)—Points D, D′ are taken on the base of the triangle ABC, and equal perpendiculars, DP, D′P′, are let fall upon the sides AB, AC. Similar lines are drawn for the other sides. Find the equation to the conic through the six points 63

15368. (Professor Nanson.)—If lines drawn through the vertices of a triangle of area Δ divide the angles into the segments α, α'; β, β'; γ, γ', and form a triangle of area Δ', then

$$\frac{\Delta'}{\Delta} = \frac{(\sin\alpha\sin\beta\sin\gamma - \sin\alpha'\sin\beta'\sin\gamma')^2}{\sin(\beta+\gamma')\sin(\gamma+\alpha')\sin(\alpha+\beta')\sin(\alpha+\alpha')\sin(\beta+\beta')\sin(\gamma+\gamma')}$$

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15369. (S. C. Bagchi, B.A.)—ABC is a triangle; a triangle DEF is formed by joining the middle points of AB, BC, CA respectively. FG is the bisector of the angle DFE, meeting DE at G. If AG is perpendicular to DF, prove that FG is the diagonal of a square whose side is equal to DG 85

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15372. (Lt.-Col. Allan Cunningham, R.E.) — Prove that 13 is a 2-ic residue of every *prime* Fermat's number $F_n = 2^{2^n}+1 > 5$. Show how this property could be used as a test of *primality* of F_n 71

15373. (R. W. D. Christie.)—For all primes of form $4m+1$, prove $r^{4m}-r^{4m}+2 \equiv 0 \pmod{p}$ (r a prime root), and give an example of its utility for factorization. *E.g.*, $12053^6-12053^4+2 \equiv 0 \pmod{12301}$ 104

15375. (Professor Nanson.)—Prove that

$$(a_1X_2Y_3Z_4)(f_1X_2Y_3Z_4)+(b_1X_2Y_3Z_4)(g_1X_2Y_3Z_4)+(c_1X_2Y_3Z_4)(h_1X_2Y_3Z_4)=0$$

where

$$X_p,\ Y_p,\ Z_p = h_py-g_pz+a_pw,\ f_pz-h_px+b_pw,\ g_px-f_py+c_pw \quad 74$$

15376. (Professor Neuberg.)—Trouver $\int \tan x \tan 2x \tan 3x\, dx$ 57

15377. (R. Chartres.) — Find the mean value of Z^n when Z is the distance between two random points (1) in a circle, (2) in a sphere; and in the latter case, if $M(Z^n) = p_n a^n$, find m and n (i.) when $p_n = p_m$, (ii.) when $p_n = 2p_m$, a being the radius of the sphere 78

15383. (A. M. Nesbitt, M.A.)—If $\alpha+\beta+\gamma$ be a multiple of π, and if $x^r \sin r\alpha + y^r \sin r\beta + z^r \sin r\gamma$ vanishes when $r=1$ and when $r=2$, it vanishes for any positive integral value of r 98

15384. (R. Tucker, M.A.)—ABC is a triangle. On AB, AC are described segments of circles containing angles equal to 2B, 2C respectively, and on BC a segment BOC, O the circum-centre. These circles intersect in P, the co-ordinates of which are

$$\frac{\alpha}{2\cos A} = \frac{\sin(2C-B)}{\sin 2C}\beta = \frac{\sin(2B-C)}{\sin 2B}\gamma.$$

Points Q, R are got in a similar manner from the other sides. Does the circle round PQR pass through O? 109

15388. (H. Bateman.) — S, S', and C are the foci and centre of a fixed conic touching four lines; T, T', and O the foci and centre of a variable conic touching the four lines. Prove that ST.S'T', S'T.S'T', and CO are to one another in constant ratios 105

15392. (Robt. W. D. Christie.)—The sum of the series

$$1+\frac{4}{3}+\frac{11}{3^2}+\frac{29}{3^3}+\ldots+\frac{a_n}{3^{n-1}} \text{ ad inf.}$$

is thrice a square integer $[a_n = 3a_{n-1}-a_{n-2}]$ 120

15395. (Robt. W. D. Christie.) — Let there be two quadratics $Q_1 = x^2+ax+b$, $Q_2 = x^2+cx+d$, and let operative multiplication be performed thus (by Major Ross's method):—$S = [Q_1]Q_2$, $\sigma = [Q_2]Q_1$, and $s = Q_1Q_2$ by ordinary algebra. Prove that

$$S+\sigma-2s \equiv \{(a-c)^2+a+c\}x^2+\{(b-d)(a-c)+ac\}.2x + (b-d)^2+b(c+1)+d(a+1)$$

for all integral values of the letters a, b, c, d, x. *E.g.*, $x=2$, $a=11$, $b=13$, $c=3$, $d=5$; then $S=403$, $\sigma=1643$, $s=39\times15$, and consequently $403+1643-1170 = 876 = 78\times4+97\times4+176$ 96

15396. (C. M. Ross.)—If

$$a^2y^2+b^2x^2 = a^2b^2,\quad a^2y(\lambda-x) = b^2x(\mu-y) \text{ ..... (i., ii.)},$$
$$y(a^4y^2+b^4x^2) = a^2b^4(y-\mu) \text{ ..... (iii.)},$$

prove that $a^{\frac{2}{3}}\lambda^{\frac{2}{3}}+b^{\frac{2}{3}}\mu^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$ 96

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15401. (Professor L. Catalá.) — Décrire une circonférence qui soit tangente à une droite ou à une circonférence en un point donné et qui passe par le point d'intersection inaccessible de deux droites données 83

15405. (J. Macleod, H.M.I.S., I.S.O.)—From the point O a tangent OQ is drawn to the point Q of the hyperbola whose focus is S. A circle is described about the triangle OQS, and in OQ a point P is taken such that QS : QO = SP : PO. SP being produced to meet the circle in X, at X in OX the angle OXR (R being on the circle) is formed equal to the angle OQS. Prove that RX = the focal distance of Q and that the focus S′ lies on QR 79

15406. (Professor Nanson.)—Show that the tangent of the angle between the axes of two parabolas is the harmonic mean of the tangents of the angles made by the tangents to either curve at the four common points with the axis of that curve 117

15412. (Professor Nanson.) — If the second differences of a series with an odd number of terms are all positive, show that the arithmetic mean of the odd terms is greater than the arithmetic mean of the even terms 95

15415. (Professor Neuberg.) — Une parallèle au côté BC d'un triangle ABC coupe les côtés AB, AC en B′, C′. Comment faut-il mener cette droite pour que les volumes engendrés par le triangle AB′C′ et par le trapèze B′C′CB tournant autour de B′C′ soient dans un rapport donné r? 101

15422. (A. Ensor.)—P and Q are taken at random on a given line AB. Find the mean of

(1) $AP.PQ.QB$, (2) $\Sigma AP.PQ$, (3) $AP^2+PQ^2+QB^2$ 98

15423. (*Revised.*) (S. C. Bagchi, B.A.)—A rod DE, of variable length, meets two others AB, BC, including a right angle at B. The triangle DBE so formed is held with its plane inclined (in a direction which shall be *definite*) to that of a screen illuminated by a parallel pencil of direct incidence. If DE be moved so as to be always at the same distance from B, prove that the locus of the foot of one of the symmedians of the shadow of BDE is an elliptic arc 110

ERRATA.

P. 28 (Question 8625), in last line but two of Solution, *for* "(M_1-M_r)" *read* "(L_2-L_r)."

P. 29, last line but one, *add* "$)$" at end of left-hand member of equation.

P. 35 (Question 15133, Solution II.), *for* "C" *read* "c" everywhere.

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

15284. (J. J. BARNIVILLE, B.A.)—Having $u_n = u_{n-2} + u_{n-3}$, prove that

$$\frac{1}{2.3.4.9} - \frac{1}{3.4.5.12} + \frac{1}{4.5.7.16} - \frac{1}{5.7.9.21} + \ldots = \frac{1}{280},$$

$$\frac{2.5.11}{2.4.7.9} + \frac{3.7.14}{2.5.9.12} + \frac{4.9.19}{3.7.12.16} + \frac{5.12.25}{4.9.16.21} + \ldots = \frac{25}{21}.$$

Solution by Professor SIRCOM.

From the relations $u_3 = u_1 + u_0$, ..., we have

$$u_1u_5 + u_0u_2 = u_1(u_3 + u_2) + u_0u_1 = u_3(u_2 + u_1) = u_3u_4, \ \ldots;$$

whence
$$\frac{1}{u_1u_3u_4u_5} + \frac{1}{u_2u_4u_5u_6} = \frac{u_2u_6 + u_1u_3}{u_1u_2u_3u_4u_5u_6}$$

$$= \frac{1}{u_1u_2u_3u_6} - \frac{1}{u_2u_4u_5u_6} - \frac{1}{u_3u_5u_6u_7} = -\frac{1}{u_2u_3u_4u_7},$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots;$$

whence, adding, everything vanishes except the first term, and the sum of the series is

$$\frac{1}{u_1u_3u_4u_5} = \frac{1}{280}.$$

We have $u_5 - u_0 = u_2 + u_3 - u_0 = u_2 + u_1 = u_4 = u_6 - u_3$;

whence
$$\frac{u_1u_4(u_6 + u_0)}{u_0u_3u_5u_6} = \frac{u_1u_4}{u_0u_3u_5} + \frac{u_1u_4}{u_3u_5u_6} = \frac{1}{u_0} - \frac{1}{u_5} - \frac{u_4}{u_3u_5} + \frac{u_1u_4}{u_3u_5u_6}$$

$$= \frac{1}{u_0} - \frac{1}{u_5} - \frac{1}{u_3} + \frac{1}{u_6},$$

since $u_6 - u_1 = u_5$; whence, adding, the sum of the series is

$$\frac{1}{u_0} + \frac{1}{u_1} + \frac{1}{u_2} - \frac{1}{u_5} = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} - \frac{1}{7} = \frac{25}{21}.$$

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15241. (Professor SANJÁNA, M.A.)—In a plane triangle ABC, (1) if the sides a, b include an angle $\cos^{-1}\sqrt{(ab)}/[a-\sqrt{(ab)}+b]$, prove that $\tan A/\tan B = -(a^3/b^3)^{\frac{1}{2}}$, so that one of the remaining angles is obtuse; (2) if $\tan C$, $\tan A$, $\tan B$ be in geometric progression, the common ratio is b^2/c^2, and A is greater than $\frac{1}{3}\pi$; (3) if $\tan\frac{1}{2}C$, $\tan\frac{1}{2}A$, $\tan\frac{1}{2}B$ be in geometric progression, the common ratio is b/c, and A is less than $\frac{1}{3}\pi$. [The triangle in (1) is formed by the Earth with the Sun and a planet at its stationary point; the triangle in (2) has the line joining its BROCARD points perpendicular to BC.]

Solution by T. SAVAGE, B.A.

(1) $$\frac{\tan A}{\tan B} = \frac{c^2+a^2-b^2}{b^2+c^2-a^2}.$$

But $$c^2 = a^2+b^2-2ab\cos C = a^2+b^2-\frac{2\sqrt{(a^3b^3)}}{a-\sqrt{(ab)}+b}.$$

By substitution and reduction, we obtain $\dfrac{\tan A}{\tan B} = -\dfrac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}$.

(2) $$\frac{\tan A}{\tan C} = \frac{a^2+b^2-c^2}{b^2+c^2-a^2}, \quad \frac{\tan B}{\tan A} = \frac{b^2+c^2-a^2}{c^2+a^2-b^2};$$

therefore $$\frac{a^2+b^2-c^2}{b^2+c^2-a^2} = \frac{b^2+c^2-a^2}{c^2+a^2-b^2} = \frac{b^2}{c^2}.$$

Also $$\frac{\tan C+\tan B}{1-\tan C\tan B} = \tan(B+C) = -\tan A;$$

therefore $$\frac{\tan C}{\tan A}+\frac{\tan B}{\tan A} = -1+\tan^2 A;$$

therefore $c^2/b^2+b^2/c^2+1 = \tan^2 A$; therefore $3 < \tan^2 A$, that is $\frac{1}{3}\pi < A$.

(3) $$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}C} = \frac{s-c}{s-a}, \quad \frac{\tan\frac{1}{2}B}{\tan\frac{1}{2}A} = \frac{s-a}{s-b};$$

therefore $$\frac{s-c}{s-a} = \frac{s-a}{s-b} = \frac{b}{c}.$$

Also $$\cot\tfrac{1}{2}A = \tan\tfrac{1}{2}(B+C) = \frac{\tan\frac{1}{2}B+\tan\frac{1}{2}C}{1-\tan\frac{1}{2}B\tan\frac{1}{2}C};$$

therefore $$\cot\tfrac{1}{2}A-\tan\tfrac{1}{2}A = \tan\tfrac{1}{2}B+\tan\tfrac{1}{2}C;$$

therefore $\cot^2\frac{1}{2}A = 1+c/b+b/c > 3$. Hence $A < \frac{1}{3}\pi$.

Mr. R. TUCKER, M.A., solves the second and third parts in the following manner:—

(ii.) $\tan B\tan C = \tan^2 A$; therefore $4b^2c^2\cos^2 A = 4a^2bc\cos B\cos C$,

i.e., $$(b^2+c^2-a^2)^2 = (a^2+c^2-b^2)(a^2+b^2-c^2);$$

whence $$b^4+c^4 = a^2(b^2+c^2) \quad \text{(A)},$$

and $$b^2(b^2+c^2-a^2) = c^2(a^2+b^2-c^2),$$

i.e., $$\frac{b^2}{c^2} = \frac{a\cos C}{c\cos A} = \frac{\tan A}{\tan C} = f;$$

therefore, &c. Now $\tan A+\tan B+\tan C = \tan A\tan B\tan C$;

therefore $$\tan A(1+f+1/f) = \tan^3 A;$$

therefore $\tan A = \sqrt{(1+f+1/f)} > \sqrt{3}$; therefore $A > \frac{1}{3}\pi$.

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Again, take the projections of ΩB and of Ω′C on BC; their sum

$$= (ac^2 \cos\omega)/\lambda + (ab^2 \cos\omega)/\lambda = [a(b^2+c^2)\,k]/2\lambda^2$$
$$= a/2\lambda^2\,[(b^2+c^2)\,a^2 + (b^2+c^2)^2] = \text{by (A)},\ a;$$

therefore, &c.; or this result may be got by proving the line ΩΩ′,

$$\Sigma bc\,(a^4 - b^2c^2) = 0,$$

to be perpendicular to BC (by A).

(iii.) $\tan\frac{1}{2}B \tan\frac{1}{2}C = \tan^2\frac{1}{2}A$; therefore $\frac{s-a}{s} = \frac{s-b\,.\,s-c}{s\,.\,s-a}$;

therefore $(s-a)^2 = s-b\,.\,s-c$, *i.e.*, $(b+c-2a)\,s = bc - a^2$;

therefore $b^2+c^2 = a\,(b+c)$, *i.e.*, $\frac{b}{c} = \frac{b+c-a}{a-c+b} = \frac{s-a}{s-c} = \frac{\tan\frac{1}{2}C}{\tan\frac{1}{2}A} = f$;

therefore, &c. Now

$$1 = \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C + \tan\tfrac{1}{2}C \tan\tfrac{1}{2}A + \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B$$
$$= \tan^2\tfrac{1}{2}A\,(1 + f + 1/f) > 3\tan^2\tfrac{1}{2}A;$$

therefore $A < \frac{1}{3}\pi$.

15343. (A. M. Nesbitt, M.A.)—If

$$x+y+z = 0 \quad \text{and} \quad ax^2 + by^2 + cz^2 = 0,$$

prove that, if λ^2 be a certain rational function of a, b, c,

$$\left\{(a+\lambda)^n - (a-\lambda)^n\right\}x^n + \left\{(b+\lambda)^n - (b-\lambda)^n\right\}y^n + \left\{(c+\lambda)^n - (c-\lambda)^n\right\}z^n = 0.$$

Solution by R. F. Davis, M.A.

If $f(t, \lambda) = \left\{1 + (a+\lambda)xt\right\}\left\{(1 + (b+\lambda)yt\right\}\left\{1 + (c+\lambda)zt\right\}$, it will be found that $f(t, -\lambda) \equiv f(t, \lambda)$, if

$$(a+\lambda)(b+\lambda)(c+\lambda) = (a-\lambda)(b-\lambda)(c-\lambda) \quad \text{or} \quad \lambda^2 = -(bc+ca+ab).$$

Therefore $f(t, \lambda)/f(t, -\lambda) \equiv 1$. [For $\Sigma ax(y+z) = 0$, and therefore

$$\Sigma(b+c)yz = 0.]$$

If now we expand $\log f(t, \lambda) - \log f(t, -\lambda)$ in ascending powers of t and equate the coefficient of t^n to zero, we have the result as stated.

6366. (G. J. Griffiths, M.A.)—Prove that the sum of the squares of the reciprocals of all numbers which are not divisible by the square of any prime is $15\pi^{-2}$.

Solution by Professor Nanson.

If S is the required sum, a, b, c, ... the successive prime numbers 2, 3, 5, ..., and $s_r = \Sigma n^{-r}$, it is clear that

$$S = s_2(1-1/a^4)(1-1/b^4)(1-1/c^4)\ldots$$
$$= s_2\left\{(1 + 1/a^4 + 1/a^8 + \ldots)(1 + 1/b^4 + \ldots)(1 + 1/c^4 + \ldots)\ldots\right\}^{-1}$$
$$= s_2/s_4 = \tfrac{1}{6}\pi^2 \div \tfrac{1}{90}\pi^4 = 15/\pi^2.$$

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10542. (W. J. GREENSTREET, M.A.)—Three men A., B., C. play a match. They draw lots to see which pair plays first, and the winner of the first game plays the third. A., winning the second game, wins the match. Show the chance of A. and B. having played the first game is $\frac{5}{18}$, when the skills of A., B., C. are as 4 : 3 : 2. Find the chance A. and B. did not play together.

Solutions (I.) *by* FRANCES E. CAVE; (II.) *by* A. M. NESBITT, M.A.

(I.) A. wins either by winning both games, or by beating the winner of first game. Therefore chance that A. wins the match

$$= \frac{1}{3}\left\{\frac{4}{7}\cdot\frac{4}{6} + \frac{4}{7}\cdot\frac{4}{6} + \frac{2}{5}\cdot\frac{4}{6} + \frac{3}{5}\cdot\frac{4}{7}\right\} = \frac{4^2\times 18}{3.5.6.7}.$$

Chance that A. beats B. and then C. $= \frac{1}{3}\cdot\frac{4}{7}\cdot\frac{4}{6}$.

Therefore, when it is known that A. wins, chance that A. and B. played first

$$= \frac{4^2}{3.7.6} \div \frac{4^2.18}{3.5.7.6} = \frac{5}{18}.$$

Chance that A. and B. did not play together

$$= \frac{1}{3}\cdot\frac{2}{5}\cdot\frac{4}{6} \div \frac{4^2.18}{3.5.7.6} = \frac{7}{36}.$$

(II.) (1) The chance that A. beats B. and then beats C. is $\frac{8}{21}$; (2) that A. beats C. and then B is $\frac{8}{21}$; (3) that A. beats B., who has previously beaten C., is $\frac{12}{35}$; (4) that A., beats C., who has previously beaten B., is $\frac{4}{15}$.

Hence the chance that (1) is the actual fact is $\frac{8}{21}/(\frac{16}{21}+\frac{12}{35}+\frac{4}{15}) = \frac{5}{18}$.

Also, as (4) represents the only way in which A. could have won without meeting B., the chance that (4) is the actual fact is $\frac{4}{15}+\frac{4.8}{35} = \frac{7}{36}$.

8625. (A. GORDON.)—Show that, if six points are taken on the sides of a tetrahedron, then the four spheres similar to that one through A and the three points on AB, AC, AD will meet in a point.

Additional Solution by A. M. NESBITT, M.A.

[The following is a proof of Question 8625, which may also, by a process that appears sufficiently obvious, be extended to space of more than three dimensions.]

Let the four spheres be denoted by Σ_1, Σ_2, Σ_3, Σ_4, and let $L_r = 0$ be the equation to the radical plane of Σ_r and the circumscribing sphere $S = 0$. Then we may put $\Sigma_r = S + IL_r$, where $I = 0$ is the plane at infinity.

Now, if the coordinates be proportional to the perpendiculars on the four faces of the tetrahedron, we have

$$S \equiv a\beta\gamma + b\gamma\alpha + c\alpha\beta + l\delta\alpha + m\delta\beta + n\delta\gamma, \quad 1 = A\alpha + B\beta + C\gamma + D\delta,$$

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A, B, C, D being the areas of the four faces, and a, b, c, l, m, n the lengths of the six edges. Let us further take

$$L_1 \equiv q_1\beta + r_1\gamma + s_1\delta, \quad L_2 \equiv p_2\alpha + r_2\gamma + s_2\delta, \quad L_3 \equiv p_3\alpha + q_3\beta + s_3\delta,$$
$$L_4 \equiv p_4\alpha + q_4\beta + r_4\gamma.$$

Expressing that Σ_2 and Σ_3 meet BC in the same point, we get

$$(a + Br_2)/Br_2 = Cq_3/(a + Cq_3),$$

which yields $a + Br_2 + Cq_3 = 0.$

Similarly, $b + Cp_3 + Ar_1 = 0, \quad c + Aq_1 + Bp_2 = 0,$

$$l + Dp_4 + As_1 = 0, \quad m + Dq_4 + Bs_2 = 0, \quad n + Dr_4 + Cs_3 = 0.$$

From these we get $S + A\alpha L_1 + B\beta L_2 + \ldots + \ldots = 0;$

so that $\Sigma_r \equiv A\alpha(L_1 - L_r) + B\beta(M_1 - M_r) + \ldots + \ldots \quad (r = 1, 2, 3, 4).$

The four spheres therefore pass through the point given by

$$L_1 = L_2 = L_3 = L_4.$$

15279. (Professor Sanjána, M.A.)—Two distinct sections of a cone are made by planes perpendicular to an axial plane, so as to have the same eccentricity and a common vertex; prove that their *latera recta* vary as their major axes. [Geometrical proof preferred.]

Solution by A. M. Nesbitt, M.A.

The minor axis is a mean proportional between the major axis and the latus rectum. Hence, if the eccentricity be fixed, which implies that the ratio of the minor axis to the major axis is fixed, the minor axis is proportional to the latus rectum; and therefore so also is the major axis.

15148. (R. W. D. Christie.)—Prove

$(r^{\phi(a)} - 1)^n = 0 \bmod a_1a_2a_3 \ldots a_n$, where $\phi(a_1) = \phi(a_2) = \ldots = \phi(a_n)$.

Solution by Lt.-Col. Allan Cunningham, R.E.

It is known that $r_n^{\phi(a_n)} - 1 \equiv 0 \pmod{a_n}.$

Hence $\Pi(r_n^{\phi(a_n)} - 1) \equiv 0 \pmod{a_1a_2a_3 \ldots a_n},$

which proves the theorem required.

Note by the Proposer.

We have $r^{\phi(a)} = 1 \bmod a$ and $r^{\phi(b)} = 1 \bmod b$; therefore

$$r^{\phi(a)+\phi(b)} - (r^{\phi(a)} + r^{\phi(b)}) = -1 \bmod ab.$$

Also $r^{\phi(ab)} = 1 \bmod ab$; therefore $r^{\phi(ab)} + r^{\phi(a)+\phi(b)} = r^{\phi(a)} + r^{\phi(b)} \bmod ab.$

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Let $\phi(a) = \phi(b)$; then $r^{\phi(ab)} + (r^{\phi(a)} - 1)^2 = 1 \bmod ab$; therefore

$$(r^{\phi(a)} - 1)^2 = 0 \bmod ab;$$

and generally $(r^{\phi(a)} - 1)^n = 0 \bmod a_1 a_2 \ldots a_n$ with the given condition.

Note.—Care must be taken, *e.g.* $7^{40} = 1 \bmod 132 = 1 \bmod 75$, but not $= 1 \bmod 75 \cdot 132$. However $(7^{40} - 1)^2 = 0 \bmod 75 \cdot 132$ as per the theorem.

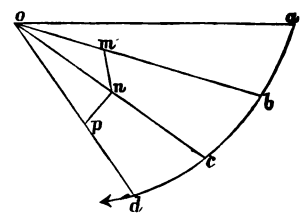
9448. (B. F. Finkel.)—A dog, on the top of a conical hill whose base is 200 feet in radius and height 100 feet, is in pursuit of a fox at the base which keeps running around the hill at the base; find how far the dog must run to catch the fox, if the dog runs 6 feet while the fox runs 5 feet and the dog runs directly towards the fox at all times.

Solutions (I.) *by* B. C. Wallis, F.C.P.; (II.) *by the* Proposer.

(I.) $abcd \ldots$ = path of fox,

$ab = bc = cd = k$,

k subtends an angle of 2° at o at the apex of the cone, oa = slant height = 223·6 feet.



When the fox is at b, c, d, ..., the dog is at m, n, p, ... While the fox goes k feet the dog goes

$$om = mn = np = l = \tfrac{6}{5} k.$$

k subtends an angle 2° 14′ 40″ at centre of base; therefore chord $ab = 7 \cdot 83\frac{5}{8}$ feet, arc $k = 7 \cdot 83558$ feet; difference = ·00225 foot, which is negligible.

Let om, on, op, ... be represented by A, B, C, ... respectively. By using the formula $\cos a/2 = \sqrt{[s(s-a)]}/bc$ we get

$$\cos^2 1° = [(om + \tfrac{1}{2} on)(\tfrac{1}{2} on)]/om\,on = [(l + \tfrac{1}{2}B)\tfrac{1}{2}B]/lB,$$

which gives B, and $\cos^2 1° = [\frac{1}{2}(B + C + l)\frac{1}{2}(B + C - l)]/BC$, which gives

$$C = \sqrt{[l^2 - 4B^2 \cos^2 1 (1 - \cos^2 1)]} - B(1 - 2\cos^2 1)$$

and

$$D = \sqrt{[l^2 - 4C^2 \cos^2 1 (1 - \cos^2 1)]} - C(1 - 2\cos^2 1), \ldots,$$

and

$$A = l.$$

From this (using the slide rule) the following results are obtained in feet:—

A = 9·4,	H = 74,	O = 134·55,	V = 186·45,
B = 18·75,	I = 82·93,	P = 142·54,	W = 192·99,
C = 28·08,	J = 91·75,	Q = 150·36,	X = 199·32,
D = 37·04,	K = 100·57,	R = 157·98,	Y = 205·41,
E = 46·67,	L = 109·22,	S = 165·43,	Z = 211·25,
F = 55·88,	M = 117·87,	T = 172·64,	A′ = 216·79,
G = 65·02,	N = 126·32,	U = 179·67,	B′ = 222·06.

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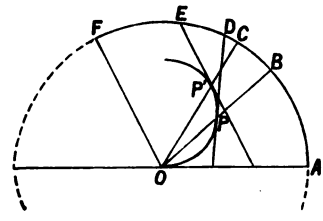
From this it is obtained that the dog catches the fox at $28{\cdot}345 \times k$ feet round the base, $28{\cdot}345 \times l$ feet down the hill, that is to say, the fox goes 222 feet and the dog goes 266 feet approximately. The error involved here is very small, and cannot exceed $28{\cdot}345 \times {\cdot}00225$ feet = ·0637... foot.

This result is substantially the same as that obtained by taking the angle *aob* 4° and 6° and 8°; and is in close agreement with the result obtained by tracing the curve with cotton on a cone composed of modelling-clay, with proportional dimensions.

(II.) Conceive the surface of the cone to be spread out on a plane. Let O be the vertex of the cone when thus spread out and ABCDEF its circumference. This surface may be repeated a sufficient number of times to complete the race. We may conceive the surface as being an infinitely thin membrane and allowed to overlap; then when the race has continued on the second round about the point O we may conceive the dog in the surface beneath, and so on for any number of rounds, in each round the position of the dog being in the surface beneath the previous surface. In this way the problem is reduced to that of the fox running in the circumference of a circle radius $100\sqrt{5}$ feet and the dog starting from the centre and running directly towards the fox at all times.



In the figure, let P and P′ be two consecutive positions of the dog and D and E the corresponding positions of the fox. Let $V_d = 6$ feet, the velocity of the dog; $V_f = 5$ feet, the velocity of the fox; and $n = V_d/V_f$. Let (x, y) be the co-ordinates of the point P and (ξ, η) the co-ordinates of the corresponding point D of the path of the fox. Then

$$\eta - y = (dy/dx)(\xi - x) \quad \ldots\ldots(1)$$

is the equation of the tangent PD;

$$\eta^2 + \xi^2 = R^2 = 500 \quad \ldots\ldots(2)$$

and PP′ = n times DE, or

$$\sqrt{(dx^2 + dy^2)} = n\sqrt{(d\eta^2 + d\xi^2)} \quad \ldots\ldots(3).$$

Eliminating η and ξ from these three equations, we have for the differential equation of the path of the dog

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]\left\{\frac{nR\frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)\frac{dy}{dx}\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}}{nR\frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}\right\}^2$$

$$+ 2\left(x\frac{dy}{dx} - y\right)\frac{dy}{dx}\left\{\frac{nRx\frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)\frac{dy}{dx}\sqrt{\left[1 + \left(\frac{dy}{dx}\right)\right]}}{nR\frac{d^2y}{ax^2} + \left[1 + \left(\frac{dy}{ax}\right)^2\right]^{\frac{3}{2}}}\right\}$$

$$+ R^2 - \left(x\frac{dy}{dx} - y\right)^2 = 0.$$

This differential equation has never been integrated, so far as I know.

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15336. (R. Chartres.)—Express

$$\int_0^{\pi}\int_0^{\theta} \sin^n\theta \sin^n\phi \sin(\theta-\phi)\, d\theta\, d\phi$$

as a single integral of $\sin\theta$ between the limits 0 and $\frac{1}{2}\pi$. Could

$$\int_0^{\pi}\int_0^{\theta} \sin^{n+2}\theta \sin^{n+2}\phi \sin^n(\theta-\phi)\, d\theta\, d\phi$$

be similarly expressed?

Solution by the Proposer.

Expanding $\sin(\theta-\phi)$,

$$\int_0^{\pi}\int_0^{\theta} \sin^n\theta \sin^n\phi \sin(\theta-\phi)\, d\theta\, d\phi = \frac{2}{n+1}\int_0^{\pi} \sin^{2n+2}\theta\, d\theta$$
$$= \frac{4}{n+1}\int_0^{\frac{1}{2}\pi} \sin^{2n+2}\theta\, d\theta.$$

Similarly, by parts,

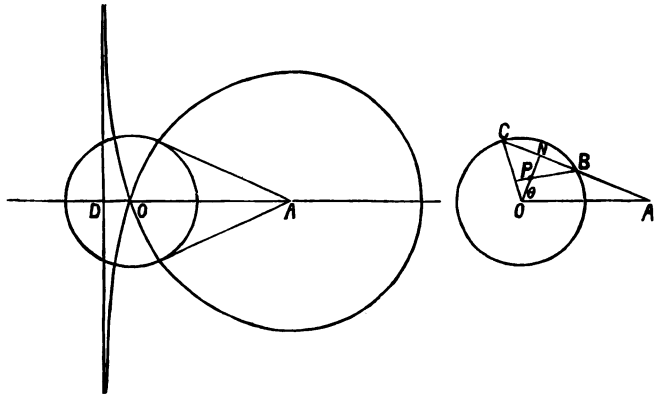
$$\int_0^{\pi}\int_0^{\theta} \sin^p\theta \sin^q\phi\, d\theta\, d\phi = \frac{q-1}{q}\,\frac{q-3}{q-2}\ldots\int_0^{\pi} \sin^p\theta\, d\theta$$
$$= 2\int_0^{\frac{1}{2}\pi} \sin^p\theta\, d\theta \times \int_0^{\frac{1}{2}\pi} \sin^q\theta\, d\theta.$$

15077. (Professor Cochez.)—On donne une circonférence O et un point A, par lequel on mène une sécante variable coupant la courbe en B et C. Lieu de l'orthocentre du triangle BOC.

Solutions (I.) *by* James Blaikie, M.A., *and* Frank Porter, B.A.; (II.) *by* R. Tucker, M.A.; (III.) *by* Lionel E. Reay, B.A.

(I.) Draw ON perpendicular to BC and BP perpendicular to OC, meeting ON in P. The locus of P is required. Let

$$OP = r, \quad \angle POA = \theta, \quad OA = a, \quad OB = b;$$



then, since P is orthocentre,

$$ON \cdot PN = BN \cdot NC = BN^2;$$

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therefore $a\cos\theta\,(a\cos\theta - r) = b^2 - a^2\cos^2\theta$,

which gives $r = 2a\cos\theta - b^2/(a\cos\theta)$, the polar equation. Transforming to rectangular co-ordinates through O, we obtain the cubic equation $(x^2+y^2)(ax+b^2) = 2a^2x^2$, from which it is clear that $ax + b^2 = 0$ is the equation to the asymptote, which is therefore a line through D at right angles to OA, when $OD = b^2/a$, and that no part of the locus lies to the left of the asymptote. The curve cuts the circle on the line $x = b^2/a$, which is the polar of A, and also on the line $x = -b^2/2a$, and meets OA at O and also at $x = 2a - b^2/a$. The drawing shows the form of the curve, and it may be noted that the part of the curve to the right of the polar of A corresponds to positions in which the secant does not meet the circle in real points.

(II.) Let E be the orthocentre on the perpendicular OD,

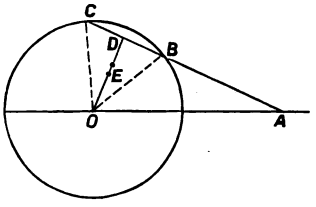
$OB = a$, $OA = c$,

$\angle AOD = \theta$, $\angle BOD = \phi$.

Then, if $OE = r$ and ρ = circum-radius of OBC,

$r = 2\rho\cos 2\phi$, $a = 2\rho\cos\phi$,

$c\cos\theta = a\cos\phi$;



whence $\dfrac{r}{a} = \dfrac{\cos 2\phi}{\cos\phi} = \dfrac{2\cos^2\phi - 1}{\cos\phi} = \dfrac{2c^2\cos^2\theta - a^2}{ac\cos\theta}$;

therefore $cr\cos\theta = 2c^2\cos^2\theta - a^2$ or $y^2(a^2+cx) = x^2(2c^2 - cx - a^2)$.

(III.) Let $OB = a$, $OA = b$, $\angle OAC = \phi$, $\angle AOB = \psi$. If P is orthocentre, $OP = r$, $AOP = \theta$; then $\theta = 90° - \phi$ and

$$r = OP = \frac{OB}{\sin OCB}, \quad \cos BOC = \frac{a}{\sin(\psi+\phi)}\cos BOC.$$

But $$\frac{OA}{OB} = \frac{\sin OBA}{\sin OAB} = \frac{\sin(\psi+\phi)}{\sin\phi} = \frac{\sin(\psi+\phi)}{\cos\theta};$$

therefore $$r = \frac{a^2}{b\cos\theta}\cos BOC.$$

But $$BOC = 180° - 2OBC = 180° - 2(\psi+\phi);$$

therefore $\cos BOC = -\cos 2(\psi+\phi) = 2\sin^2(\phi+\psi) - 1 = 2(b^2/a^2)\cos^2\theta - 1$;

therefore the locus of P is given by the equation

$$rb\cos\theta = 2b^2\cos^2\theta - a^2.$$

15297. (C. M. Ross.)—P is a point in the diameter of the semicircle ACDB, PC is drawn at right angles to AB, and PEB is a semicircle on the same side of AB. If QR be a diameter of the circle which touches PC in Q and the semicircles in D and E, show that the rectangle AB . QR is equal to the square on CP.

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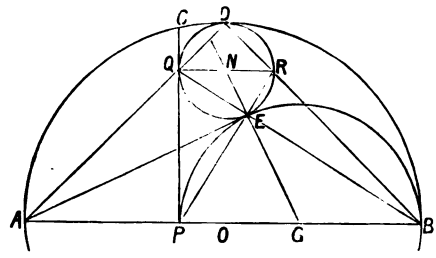
Additional Solution by H. W. Webster, M.D.



If N, O, G be the centres, DNO, NEG will be straight lines. DN/NR = DO/OB; therefore BRD is a straight line. By similar proofs DQA, REP, and QEB are straight lines. Then Q, P, B, D are cyclic points. Therefore AQ.AD = AP.AB; therefore A is on the radical axis AE. The triangles AEP, ABE are similar, and vary as their bases, or as the squares of their corresponding sides. Therefore

$$AP/AB = PE^2/BE^2 = (QE.EB)/BE^2 = QE/BE = QR/PB$$

(QR being parallel to PB and PE perpendicular to QB, ...); therefore

$$AB.QR = AP.PB = PC^2.$$

15133. (John Prescott, B.A.)—Through P, a point within the angle BAC, draw a line MPN, meeting AB and AC in M and N respectively, such that the sum of AM and AN shall be equal to a given length. Also draw the line so that the sum of AM and AN may be a minimum.

Solutions (I.) *by* A. M. Nesbitt, M.A.; (II.) *by* A. W. Poole *and* R. Tucker, M.A.; (III.) *by* Lionel E. Reay, B.A.; (IV.) *by* D. Biddle.

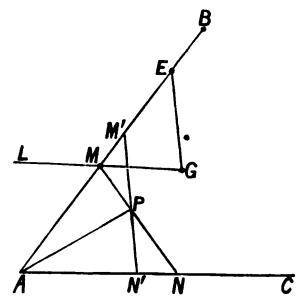
(I.) (1) Make AE equal to the given length; and make $\angle AEG = \angle PAC$, taking EG = PA. On GP describe a segment containing an angle equal to BAC, cutting AB in M, M'. Join MPN, M'PN'. Then, since

$$\angle GMP = \angle BAC,$$

GL touches the circle described about AMN; therefore

$$\angle GME = \angle LMA = \angle MNA;$$

therefore the triangles GEM, PAN are equal in all respects; therefore ME = AN. Similarly M'E = AN'.



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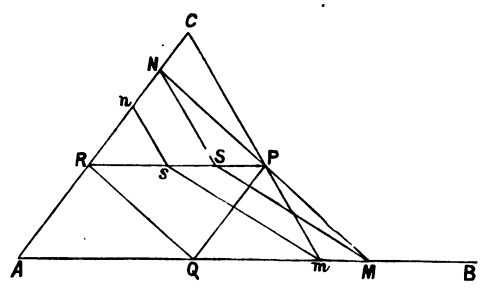
(2) Draw PQ, PR parallel to AC, AB. Take $RS^2 = AR \cdot AQ$ and draw SM parallel to QR. Join MPN, SN. Then $QM/RP = QP/RN$ therefore $RN \cdot QM = RS^2$; therefore $RN = QM$.

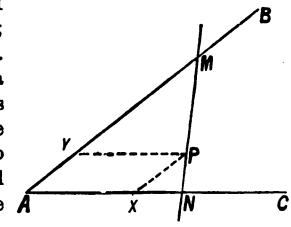


Draw *any other* transversal CP, and draw *ms*, *sn* parallel to MS, SN. Then

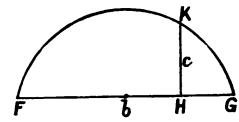
$$Qm/RP = QP/RC;$$

therefore $RN^2 = Rn \cdot RC$. Therefore $NC > Nn$, *i.e.*, $NC > Mn$ therefore $Ac + An > AN + AM$, and therefore the latter is a minimum.

(II.) Draw PY, PX parallel to AN, AM respectively. Then $MY/YP = PX/XN$; therefore $MY \cdot XN = PX \cdot PY = \text{const.}$ If then the sum of AM and AN is to be a given length, the sum of YM and XN is given, because P is a fixed point. But we know $YM \cdot XN = C^2$, say; and we also know $YM + XN = b$, say. It is required to find YM and XN. This is easily done by *Euclid* II. 14.



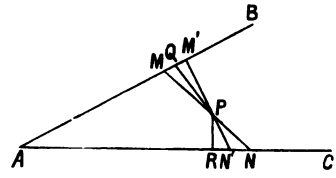
Let $FG = b$. On FG describe a semicircle and draw a straight line parallel to FG and distant C from FG. Let it cut the semicircle in K. Draw KH perpendicular to FG. Then $FH = YM$, $HG = XN$, or $FH = XN$, $HG = YM$. $AM + AN$ is a minimum when $YM + XN$ is a minimum. But $MY \cdot XN = C^2$; therefore for a minimum $YM = XN = C$. The construction is obvious.



(III.) (1) If a range of points (M) be taken on AB, and two other ranges (*n*) and (N) on AC, such that $AM + An$ = given length, and M, P, N are collinear, then

range (*n*) = range (M)
= P(M) = P(N) = (N);

therefore the ranges (*n*) and (N) are homographic, and their common point will be the point required on AC.



(2) When $AM + AN$ is a minimum.—Let MPN, M'P'N' be two positions of the line consecutive to the critical position; then

$$AM + AN = AM' + AN';$$

therefore $MM' = NN'$. Draw PQ and PR perpendicular to AB

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and AC. $\Delta PMM'/\Delta PNN' = PQ/PR$ and also $= PM^2/PN^2$ in the limit. Therefore the ratio of PM : PN is known; therefore the ratio of $\Delta AMP : \Delta APN$ is known; therefore the ratio of AM : AN is known, since the angles MAP and NAP are known. Then, if M, P, N are collinear, and n is taken so that AM : An in this known ratio, $(n) = (M) = (N)$. Therefore the required point is a common point of the homographic ranges (n) and (N).

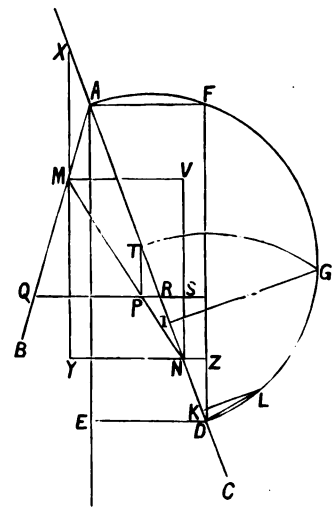
(IV.) Let AD be the given length, and $DN = x$. Bisect the angle A by AE and draw DE perpendicular to AE. Produce CA, and assume $AX = DN$; then XY, drawn parallel to AE, will cut AB in M, for $AM = AX = ND$, so that $NA + AM = AD$. In order that MN may pass through P we must have the ratio NS : SP = NV : VM, that is to say,

$$(RD - x)\cos\tfrac{1}{2}A : PR + (RD - x)\sin\tfrac{1}{2}A = (AD - 2x)\cos\tfrac{1}{2}A : AD\sin\tfrac{1}{2}A,$$

which by reduction becomes $RD - x : TD - x = AD - 2x : AD$. Let AD be taken as unity; then $x^2 - TDx = -\tfrac{1}{4}TR$; whence

$$x = \tfrac{1}{2}\{TD \pm (TD^2 - 2TR)^{\frac{1}{2}}\}.$$

Construction.—On AD describe the semicircle AFD, and from D as centre with radius TD describe the arc TG, cutting the semicircle in G. Draw GI perpendicular to AD, and make IK = 2TR. Draw KL at right angles to AD, to cut the semicircle in L, and join DL. Then $\tfrac{1}{2}(TD \pm DL) = x = DN$, the minus sign being taken in the present instance. NP produced meets AB in M, and MN is the line required. It should be possible to draw a second line M′PN′ fulfilling the conditions, M′ being lower down than M, whilst N′ is higher up than N, and this is the case when M′M = N′N = DL, the difference between the results when taking the positive and negative signs in the above formula.



As to the second part of the Question, $AM'_1 + AN'_1 = AD'$ is clearly a minimum when only one line $M'_1PN'_1$ can be drawn with that result; that is, when

$$D'L = (TD'^2 - 2TR)^{\frac{1}{2}} = 0,$$

and consequently $TD' = (2TR)^{\frac{1}{2}}$. Now $TD' = AD' - AT$, and we have the following ratio: $AD' - AT : AD' = 2TR : AD' - AT$; whence $AD' = AT + TR \pm (TR^2 + 2AT \cdot TR)^{\frac{1}{2}}$, but only the positive sign is possible. In order to find AD′ geometrically, we now take AT as unity, and on it describe a semicircle. With centre T and radius TR, we next de-

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scribe an arc cutting the semicircle in (say) G′, and draw the perpendicular G′I′ to meet AT in I′. From I′A we now mark off I′K′ = 2TR and draw K′L′ at right angles to meet the semicircle in L′. Joining TL′, we have AT + TR + TL′ = AD′. We need therefore only make RD′ = TL′ to find D′, and bisect TD′ to find N'_1; for our formula gives us

$$D'N'_1 = x = \tfrac{1}{2}TD',$$

since D′L = 0 (see above).

In the accompanying diagram, K nearly occupies the position of D′, and the mid-point of TK is nearly identical with N'_1.

14973. (H. M. Taylor, M.A., F.R.S.)—In how many ways can a party of p married couples be arranged round a table, ladies and gentlemen being seated alternately, and no man being next his wife?

Solution by the Proposer.

[Extracted, by kind permission, from the *Messenger of Mathematics*, New Series, No. 376, August, 1902.]

The following paper is the result of an attempt to solve the problem:—

"To find the number of different ways in which a party of n married couples can be seated round a table, the ladies and gentlemen being seated alternately, and no man being next his wife."

We shall consider here only arrangements in which the ladies and gentlemen are seated alternately, and we shall call those in which no man is next his wife *perfect* arrangements; those in which one man and one only is next his wife *singly-imperfect* arrangements; and those in which two men and two only are next their wives *doubly-imperfect* arrangements.

Let us denote the n couples by the letters Aa, Bb, Cc, Dd, ..., Zz, the ladies being distinguished by small letters.

Let $f(n)$ be the number of ways of seating the n ladies so as to form perfect arrangements, the n gentlemen being assumed already seated in some definite, say alphabetical, order. Then the total number of ways of seating the n couples, taking into consideration all possible arrangements of the n gentlemen among themselves, will be $(n-1)!f(n)$.

Let $\phi(n)$ be the number of ways of seating the ladies so as to form singly-imperfect arrangements, the n gentlemen being assumed, as before, seated in alphabetical order, one named lady only being next her husband, and she on a definite side of him, right or left; and let $\psi(n)$ be the number of ways of seating the ladies so as to form doubly-imperfect arrangements, one definite lady being next her husband on a definite side of him, and some one other lady and only one being next her husband, the side being indefinite.

If in any perfect arrangement of the n couples Aa, Bb, Cc, ... we interchange b with the lady between B and C, we shall get an arrangement of the n couples which may be either singly-imperfect or doubly-imperfect, since b is necessarily next B, and the lady whom b has been

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changed with may be or may not be next her husband. If from such an arrangement we withdraw the couple Bb, we obtain an arrangement of the $n-1$ couples Aa, Cc, Dd, ... which may be either perfect, singly-imperfect, or doubly-imperfect. Conversely, it follows that every perfect arrangement of the n couples can be obtained from some arrangement of the $n-1$ couples Aa, Cc, Dd, ... perfect, singly-imperfect, or doubly-imperfect by introducing the couple Bb between A and C and then interchanging b with some other lady.

From any perfect arrangement of the $n-1$ couples such as Ax C... Lc M... we can obtain a perfect arrangement of the n couples by introducing the couple Bb between x and C and then interchanging b with any other lady except x or c, viz. in $n-3$ ways.

Hence from perfect arrangements of $n-1$ couples we obtain $(n-3)f(n-1)$ different perfect arrangements of the n couples.

From a singly-imperfect arrangement of the $n-1$ couples such as Ac C... we can by introducing the couple Bb between c and C and then interchanging b with any other lady except c obtain $n-2$ different perfect arrangements of the n couples.

In this way we can obtain $(n-2)\phi(n-1)$ different perfect arrangements of the n couples.

From a singly imperfect arrangement such as either Aa C... or Ax Cc... we cannot by introducing the couple Bb and then interchanging b with a or c respectively obtain a perfect arrangement of the n couples. If, however, we take any one of the remaining $2n-5$ forms of singly-imperfect arrangements Ax Cd D... ; Ax Cy Dd... ; ... ; Ax C ... Zz or Ax C...Za, we can by introducing the couple Bb between A and C and then interchanging b with the lady who is the cause of the imperfection of the arrangement obtain one perfect arrangement of the n couples.

Hence we can obtain from singly imperfect arrangements of the $n-1$ couples $(n-2)\phi(n-1)+(2n-5)\phi(n-1)$ or $(3n-7)\phi(n-1)$ different perfect arrangements.

Again, from any doubly-imperfect arrangement of the $n-1$ couples of the form Ac C ... Ll M ... we can by introducing the couple Bb between c and C and then interchanging b and l obtain a perfect arrangement of the n couples. We can therefore obtain a perfect arrangement of the n couples from a doubly-imperfect arrangement of the $n-1$ couples in $\psi(n-1)$ ways.

We have now obtained all the possible perfect arrangements, and we have not counted any one of them twice. This appears from the fact that every perfect arrangement of the n couples leads to some arrangement of the $n-1$ couples of one of the three kinds described, and that a unique form; and therefore in the converse process no two different arrangements of the $n-1$ couples of the three kinds described could lead to the same perfect arrangement of the n couples.

We may therefore now write down the equation

$$f(n) = (n-3)f(n-1)+(3n-7)\phi(n-1)+\psi(n-1) \quad \ldots\ldots\ldots\ldots (1).$$

In a similar manner we see that singly-imperfect arrangements of the n couples of the form Ax Bb C ... can be obtained from any perfect arrangement of the $n-1$ couples of the form Ax C ... or from any singly-

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imperfect arrangement of the form $Ac\,C\ldots$ by the simple introduction of the couple $B\delta$ between x and C or between c and C respectively, and in no other way, and that all of them will be different.

Hence $$\phi(n) = f(n-1) + \phi(n-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Similarly we see that doubly-imperfect arrangements of the n couples can be obtained by the simple introduction of the couple $B\delta$ in any singly-imperfect arrangement of the $2n-3$ forms $Aa\,C\ldots$; $Ax\,Cc\ldots$; $Ax\,C\delta\,D\ldots$; $\ldots$; $Ax\,C\ldots Zz$; $Ax\,C\ldots Za$ or in any doubly-imperfect arrangement of the form $Ac\,C\ldots Ll\,M\ldots$.

Therefore $$\psi(n) = (2n-3)\,\phi(n-1) + \psi(n-1) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

By elimination of pairs of the functions f, ϕ, ψ from the equations of the form (1), (2), (3) we obtain the equations

$$nf(n+2) - (n^2+n+1)f(n+1) - (n^2+n+1)f(n) - (n+1)f(n-1) = 0 \ldots(4),$$

$$\phi(n+2) - n\phi(n+1) - n\phi(n) - \phi(n-1) = 0 \ldots(5),$$

$$(2n-1)(2n-3)\psi(n+2) - (n+1)(2n-3)^2\psi(n+1) - (n-2)(2n+1)^2\psi(n) - (2n-1)(2n+1)\psi(n-1) = 0 \ldots(6).$$

We might solve the equations (4), (5), (6) independently.

On account of the simpler form of equation (5) we choose that one for solution, although the method used is equally applicable to either of the others. By eliminating the functions $\phi(n+1), \phi(n), \ldots, \phi(5), \phi(4)$ from the equation (5) and the next $n-2$ equations of the same form we obtain as the general solution of equation (5)

$$\phi(n+2) = \begin{vmatrix} n, & n, & 1, & & \ldots & & & & & \\ -1, & n-1, & n-1, & 1, & \ldots & & & & & \\ & -1, & n-2, & n-2, & 1 & \ldots & & & & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & & & \ldots & -1, & 4, & 4, & \phi(3), & \\ & & & & \ldots & & -1, & 3, & 3\phi(3) + \phi(2), & \\ & & & & \ldots & & & -1, & 2\phi(3) + 2\phi(2) + \phi(1) & \end{vmatrix}$$

If we adopt as the values of $\phi(1), \phi(2), \phi(3)$ their actual values 1, 0, 0 respectively, we obtain the general solution for ϕ as applicable to the problem in hand.

When the function ϕ has been found the other functions will be given by the equations $$f(n) = \phi(n+1) - \phi(n)$$

and $$\psi(n) = \phi(n+1) - (n-2)\,\phi(n) + \phi(n-1).$$

The values of the functions for the first ten values of n are given in the following table :—

n	$f(n)$	$\phi(n)$	$\psi(n)$	$(n-1)!f(n)$
1	−1	1	0	−1
2	0	0	1	0
3	1	0	1	2
4	2	1	1	12
5	13	3	8	312
6	80	16	35	9600
7	579	96	211	410800
8	4738	675	1459	23,879520
9	43387	5413	11584	1749,363840
10	439792	48800	103605	159591,720960.

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15326. (V. DANIEL, B.Sc.)—A man leaving his home for a country station averages 120 yards per minute walking and 200 running, but he misses his train by 5 minutes. If he had run where he walked, and *vice versa*, he would have had 10 minutes to wait, having exceeded his former average for the whole distance by $43\frac{3}{4}$ yards per minute. How long was he on the road?

Solutions (I.) *by* Rev. T. WIGGINS, B.A., A. W. POOLE, *and* A. E. DAVIES
(II.) *by* W. E. HARRISON, A.R.C.S., *and* B. C. WALLIS, F.C.P.

(I.) Suppose him to walk x yards, and to run y yards; therefore

$$\tfrac{1}{120}x + \tfrac{1}{200}y - \tfrac{1}{200}x - \tfrac{1}{120}y = 15 \quad \text{.....(i.)}$$

or $$x - y = 4500;$$

$$[(x+y)/(\tfrac{1}{200}x + \tfrac{1}{120}y)] - [(x+y)/(\tfrac{1}{120}x + \tfrac{1}{200}y)] = 43\tfrac{3}{4} \quad \text{.....(ii.)},$$

which on reduction gives $87x^2 - 238xy - 297y^2 = 0 = (3x - 11y)(29x + 27y)$.

Combining (i.) and (ii.), $x = \frac{1}{2}(17 \,.\, 11 \,.\, 75)$, $y = \frac{1}{2}(15 \,.\, 3 \,.\, 75)$. Therefore time he was on the road, viz., $\frac{1}{120}x + \frac{1}{200}y = 60$ minutes.

Note.—The Rev. T. WIGGINS remarks that:—The equation

$$29x + 27y = 0$$

gives the answer to the following:—

A man leaving his home for a country station averages 120 yards per minute walking and 200 running. After walking a certain distance, he finds he has set off in the wrong direction and runs back again direct for the station. If he had run where he walked, and *vice versa*, the excess of the time of walking over the time of running in the first case would have been greater by 15 minutes than the excess of the time of running over the time of walking in the second case; and, if on two future occasions he accomplished the *absolute* distance of his house from the station in these excess times, the difference of the averages of the last two journeys would be $43\frac{3}{4}$ yards per minute. How long was he on the road? Answer, $29\frac{1}{16}$ minutes.

(II.) Let the distance walked $= 600x$, distance run $= 600y$. Then the time taken $= 5x + 3y$ minutes. In the second case the time $= 3x + 5y$ minutes; therefore $2x - 2y = 15$ ..... (i.).

Difference of average rates:—

$$\frac{600(x+y)}{3x+5y} - \frac{600(x+y)}{5x+3y} = \frac{175}{4},$$

$$2400(x+y)(2x-2y) = 175(3x+5y)(5x+3y) \quad \text{.....(ii.)}.$$

Let $5x + 3y = \mathrm{T}$ and $3x + 5y = t$. Then

$$8(x+y) = \mathrm{T} + t, \quad 2x - 2y = \mathrm{T} - t = 15.$$

Substituting for $(x+y)$ and $(2x-2y)$ in (ii.), we have

$$12(\mathrm{T}+t)(\mathrm{T}-t) = 7\mathrm{T}t, \quad 12\mathrm{T}^2 - 7\mathrm{T}t - 12t^2 = 0, \quad (4\mathrm{T}+3t)(3\mathrm{T}-4t) = 0.$$

The real value is given by

$$3\mathrm{T} - 4t = 0 \text{ or } 3\mathrm{T} - 4(\mathrm{T}-15) = 0, \quad \mathrm{T} = 60 \text{ minutes}.$$

NOTE.—Distance = 7875 yards.

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Note by the PROPOSER.

The equation $29x + 27y = 0$ does not legitimately lead to the solution $29\frac{11}{16}$ minutes, and Mr. WIGGINS only obtains it by arguing from the fact that y comes out negative, that the rate 200 is to be taken negative on substituting for x and y in the equation $\frac{1}{110}x + \frac{1}{100}y = T$. This one is not at liberty to do, since the value $+200$ initially produced the quadratic in x and y. The true alternative value $6\frac{3}{7}$ minutes is more clearly seen thus:

Let A and B be distances walking and running, a and b the respective rates, $t \equiv \lambda\tau$ the time on the road, τ = time excess of first over second journey, c = increase of average rate; then

$$c = (A+B)/[\tau(\lambda-1)] - (A+B)/(\lambda\tau) = (A+B)/[\lambda(\lambda-1)\tau]$$

or $$A + B = \lambda(\lambda-1)c\tau \quad \ldots\ldots (1).$$

But $$A/a + B/b = \lambda\tau, \quad A/b + B/a = (\lambda-1)\tau \quad \ldots\ldots (2, 3);$$

therefore $$(1/b + 1/a)(A+B) = (2\lambda-1)\tau \quad \ldots\ldots (4);$$

therefore $$(1/b + 1/a) = (2\lambda-1)/[\lambda(\lambda-1)c] \quad \ldots\ldots (4)\div(1);$$

therefore $$\lambda^2(c/b + c/a) - \lambda(c/b + c/a + 2) + 1 = 0,$$

i.e., $$\lambda^2(\tfrac{175}{450} + \tfrac{175}{800}) - \lambda(\tfrac{175}{450} + \tfrac{175}{800} + 2) + 1 = 0,$$

or $$7\lambda^2 - 31\lambda + 12 = 0; \quad (\lambda-4)(7\lambda-3) = 0,$$

$\lambda = \frac{3}{7}$ gives $t = 6\frac{3}{7}$ minutes.

8625. (A. GORDON.)—Show that, if six points are taken on the sides of a tetrahedron, then the four spheres similar to that one through A and the three points on AB, AC, AD will meet in a point.

Additional Solution by the Hon. G. R. DICK.

This beautiful theorem was first given, I believe, by Mr. S. ROBERTS. See his paper "On certain Tetrahedra specially related to four Spheres meeting at a Point" (*Proc. Lond. Math. Soc.*, Vol. XII.). The property is there established by means of considerations based on the results obtained by inverting the analogous theorem in two-dimensional space with respect to any point whatever. The plane theorem is this: The circles described about the triangles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ (where A, B, C are the angular points of a triangle and α, β, γ any points in the sides) meet at a point. Either theorem may be stated poristically, and this character of the three-dimensional theorem is brought out in the paper cited. The following analytical proof is submitted: it leads at once to a generalization of the theorems under consideration.

Denote the points taken in BC, CA, AB, AD, BD, CD by the letters α, β, γ, α', β', γ' and the tetrahedron ABCD as the tetrahedron of

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reference for a system of quadriplanar co-ordinates. Let the equation of the sphere circumscribing the tetrahedron be

$$S \equiv ayz + bzx + cxy + a'xw + b'yw + c'zw = 0,$$

and that of the plane at infinity $P \equiv Ax + By + Cz + Dw = 0$.

The equations of the four spheres $A\alpha'\beta\gamma$, $B\beta'\gamma\alpha$, $C\gamma'\alpha\beta$, $D\alpha'\beta'\gamma'$ are of the form following :—

$$S = P(\beta_1 y + \gamma_1 z + \delta_1 w), \quad S = P(\alpha_2 x + \gamma_2 z + \delta_2 w) \ldots\ldots\ldots (1, 2),$$

$$S = P(\alpha_3 x + \beta_3 y + \delta_3 w), \quad S = P(\alpha_4 x + \beta_4 y + \gamma_4 z) \ldots\ldots\ldots (3, 4).$$

At the point α' we have $y = 0$, $z = 0$. Therefore, since α' lies on sphere (1), we have $a'xw = \delta_1 w(Ax + Dw)$ or $x(a' - \delta_1 A) = w\delta_1 D$. Similarly, since α' lies on sphere (4), $w(a' - \alpha_4 D) = A\alpha_4 x$; whence

$$(a' = \delta_1 A)(a' - \alpha_4 D) = \alpha_4 \delta_1 AD \quad \text{or} \quad a' = \delta_1 A + \alpha_4 D.$$

Similarly, $b' = \delta_2 B + \beta_4 D$, $c' = \delta_3 C + \gamma_4 D$, $a = \gamma_2 B + \beta_3 C$, $b = \gamma_1 A + \alpha_3 C$, $c = \beta_1 A + \alpha_2 B$.

If the spheres have a common point, the equations (1), (2), (3), (4) have a common set of values : put $S = P\rho$, $x = \xi\rho$, $y = \eta\rho$, $z = \zeta\rho$, $w = \omega\rho$, and we get the equations

$$\beta_1\eta + \gamma_1\zeta + \delta_1\omega = 1, \quad \alpha_2\xi + \gamma_2\zeta + \delta_2\omega = 1 \ldots\ldots\ldots\ldots\ldots\ldots (a, b),$$

$$\alpha_3\xi + \beta_3\eta + \delta_3\omega = 1, \quad \alpha_4\xi + \beta_4\eta + \gamma_4\zeta = 1 \ldots\ldots\ldots\ldots\ldots\ldots (c, d).$$

Substituting in any one of the equations (1), (2), (3), (4), we get the condition

$$a\eta\zeta + b\zeta\xi + c\xi\eta + a'\xi\omega + b'\eta\omega + c'\zeta\omega = A\xi + B\eta + C\zeta + D\omega.$$

If this condition is satisfied, the four spheres meet at a point. Multiply the equations (a), (b), (c), (d) by $A\xi$, $B\eta$, $C\zeta$, $D\omega$ and add, thus :

$$\eta\zeta(B\gamma_2 + \beta_3 C) + \ldots = A\xi + B\eta + C\zeta + D\omega$$

or, since $a = B\gamma_2 + C\beta_3 + \ldots$,

$$a\eta\zeta + b\zeta\xi + c\xi\eta + a'\xi\omega + b'\eta\omega + c'\zeta\omega = A\xi + B\eta + C\zeta + D\omega,$$

or the necessary and sufficient condition is fulfilled.

Nowhere in the preceding is it necessary that a, b, c, a', b', c', ABCD should have any special meanings, that is to say, they may have any value ; only S then denotes a quadric passing through the vertices and P any plane whatever. The natural extension of the proposition is this : Through the points A, α', β, γ, ... are described quadrics having a given plane section : they meet at a point.

The general plane theorem is evidently this : Given any five points A, B, C, D, E ; in BC, CA, AB are taken points α, β, γ. The conics $A\beta\gamma DE$, $B\gamma\alpha DE$, $C\alpha\beta DE$ meet at a point.

The generalization for three-dimensional space is given in Mr. Roberts's paper referred to.

4759. (Professor Hudson, M.A.)—A right circular cylinder is made of elastic material attached to rigid fixed plane ends. It is distended by fluid pressure. Supposing that the tensions in the circular sections are

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regulated by HOOKE's law, obtain equations sufficient to determine completely the shape it will assume. If the pressure p be constant, prove that the meridian curve is

$$x + A = \int \frac{\frac{1}{2}py^2 + B}{\left\{(\lambda y^2/2a - \lambda y + C)^2 - (\frac{1}{2}py^2 + B)^2\right\}^{\frac{1}{2}}} dy.$$

Solution by the PROPOSER.

With the notation of BESANT's *Hydromechanics*, Art. 125, but changing x into y and z into x, we have

$$\frac{d}{ds}\left(yt\frac{dx}{ds}\right) = py\frac{dy}{ds}, \qquad t' = \frac{d}{dy}(yt).$$

If a be the radius of the cylinder, $t' = \lambda(y-a)/a$,

$$yt\frac{dx}{ds} = \tfrac{1}{2}py^2 + B, \qquad yt = \frac{\lambda y^2}{2a} - \lambda y + C;$$

therefore

$$\frac{dx}{ds} = \frac{\frac{1}{2}py^2 + B}{\lambda y^2/2a - \lambda y + C};$$

whence

$$x + A = \int \frac{(\frac{1}{2}py^2 + B)\,dy}{\left\{(\lambda y^2/2a - \lambda y + C)^2 - (\frac{1}{2}py^2 + B)^2\right\}^{\frac{1}{2}}},$$

A, B, C being constants of integration.

15339. (JAMES BLAIKIE, M.A.)—If the odd numbers be arranged in their natural order 1, 3, 5, ..., and divided into groups, such that the numbers in the groups taken in order are 1, 1, 4, 3, 9, 6, ..., $(r-1)^2$, $(r-1)\,r/2$, r^2, ..., and if every second group be struck out, beginning with the second, prove that the sum of the r^2 numbers in the r-th retained group will be r^5. This is obviously an extension of the known theorem that, if the odd numbers be divided into groups of 1, 2, 3, ... numbers, then the sum of the r numbers in the r-th group is r^3. Extend the theorem to the case of groups such that the number of numbers in the r-th group shall be r^3 and their sum r^7.

Solutions (I.) *by* Lt.-Col. ALLAN CUNNINGHAM, R.E.;
(II.) *by the* PROPOSER.

(I.) (1) The groups are of *two* kinds: (i.) those containing r^2 odd numbers in *their* r-th group, (ii.) those containing $\frac{1}{2}r(r-1)$ odd numbers in *their* r-th group.

Let n_r = total number of odd numbers in the $(r-1)$ groups of *first* kind *before* the r-th group; therefore

$$n_r = 1^2 + 2^2 + 3^2 + \ldots + (r-1)^2 = \tfrac{1}{6}r(r-1)(2r-1).$$

Let n_r' = total number of odd numbers in the $(r-1)$ groups of *second* kind *before* the r-th group of first kind; therefore

$$n_r' = 1 + 3 + 6 + \ldots + \tfrac{1}{2}r(r-1) = \tfrac{1}{2}\Sigma(r^2 - r)$$
$$= \tfrac{1}{2}\left\{\tfrac{1}{6}r(r+1)(2r+1) - \tfrac{1}{2}r(r+1)\right\} = \tfrac{1}{6}r(r+1)(r-1).$$

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Let N_r = total number of odd numbers *before* the r-th group of *first* kind; $N_r = n_r + n_r'$; therefore

$$N_r = \tfrac{1}{2}r(r-1)\{(2r-1)+(r+1)\} = \tfrac{1}{2}r^2(r-1).$$

Let a_1 = first odd number in the r-th group of first kind $= 2N_r + 1 = r^2(r-1)+1$.

Let s_r = sum of the numbers in that group; the actual numbers being $(2N_r+1)$, $(2N_r+3)$, $(2N_r+5)$, ... to r^2 terms; therefore

$$\begin{aligned} s_r &= r^2.2N_r + (1+3+5+\ldots \text{ to } r^3 \text{ terms}) \\ &= r^2.r^2(r-1) + \tfrac{1}{2}r^2\{1+(2r^2-1)\} = r^5. \end{aligned}$$

(2) To analyse the more general theorem a different notation is required; the groups will here also be of *two* kinds: (i.) those containing r^3 odd numbers in their r-th group, (ii.) those not containing r^3 odd numbers in their r-th group.

Let N_r = total number of odd numbers *up to end* of r-th group of *first* kind; S_r = sum of the above N_r numbers; N_r' = total number of odd numbers *preceding* the r-th group of *first* kind; S_r' = sum of the above N_r' numbers; n_r = total number of odd numbers in the r-th group of *first* kind; s_r = sum of the numbers in that group. Therefore

$$S_r = N_r^2,\ S_r' = N_r'^2;\quad s_r = S_r - S_r' = N_r^2 - N_r'^2 = r^7;\quad N_r - N_r' = n_r = r^3;$$

therefore $N_r + N_r' = r^4$; therefore $N_r = \tfrac{1}{2}r^3(r+1)$, $N_r' = \tfrac{1}{2}r^3(r-1)$.

Let n_r' = total number of odd numbers in the r-th group of *second* kind (which just *precedes* the r-th group of first kind). Then

$$n_r' = N_r' - N_{r-1} = \tfrac{1}{2}r^3(r-1) - \tfrac{1}{2}(r-1)^3 r = \tfrac{1}{2}r(r-1)(2r-1).$$

Hence the N_r odd numbers must be arranged in alternate groups of $n_r = r^3$ and $n_r' = \tfrac{1}{2}r(r-1)(2r-1)$ numbers. Hence the first few groups should contain 1, 3, 8, 15, 27, 42, 64, 90, 125, ... numbers.

Cor.—Similarly the odd numbers might be arranged into groups of two kinds, such that the r-th group of *first* kind should contain r^m *odd* numbers whose sum should be r^{2m+1}: the group of *second* kind (just preceding the above) will be found to contain $\tfrac{1}{2}r(r-1)\{r^{m-1} - (r-1)^{m-1}\}$ numbers.

(II.) The total number of odd numbers less than the first number in the r-th retained group is

$$\begin{aligned} \Sigma(r-1)^2 + \tfrac{1}{2}[\Sigma(r-1)r] &= \tfrac{1}{6}[(r-1)r(2r-1)] + \tfrac{1}{2}[(r-1)r(r+1)] \\ &= \tfrac{1}{3}[r^2(r-1)]. \end{aligned}$$

We have therefore to sum an arithmetical series of r^2 terms whose first term is $r^2(r-1)+1$ and whose common difference is 2, which gives r^5 as the result.

Extension.—If we form the odd numbers into groups of 1, 3, 8, 15, 27, 42, ..., $(r-1)^3$, $\tfrac{1}{2}[(r-1)r(2r-1)]$, r^3, ..., and strike out every second group beginning with the second, the sum of the r^3 numbers in the r-th retained group will be r^7.

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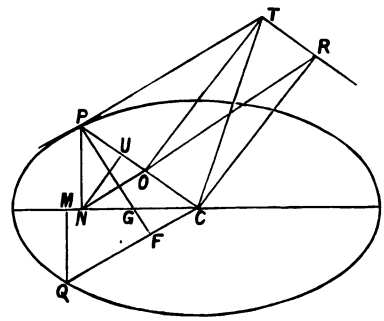
15350. (A. M. Nesbitt, M.A.)—On the tangent at P to an ellipse a point T is taken such that TP = TC; TR and TP are equally inclined to the axis, and CR is drawn perpendicular to TR. If CQ be conjugate to CP, prove that the ratio CR : CQ is constant.

Solutions (I.) *by* R. F. Davis, M.A., *and* Professor Sanjána, M.A.; (II.) *by* S. C. Bagchi, B.A.

(I.) Let PN be the ordinate of P; NU perpendicular upon CP; O the middle point of CP, so that TO is perpendicular to CP. Since (ON, OC) and (TR, TP) are equally inclined to the axis, the angles between ON, TR and OC, TP (= OC, TC) are equal. Thus N, O, R are collinear. By similar triangles,



$$CR : CT = NU : \tfrac{1}{2}CP$$

and $CT : \tfrac{1}{2}CP = CP : CF$.

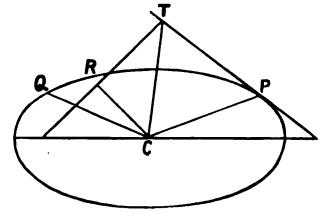
Hence

$$CR.CF = NU.CP = CN.PN.$$

But $CF . CQ = CG . CM \propto CN . PN$; and CR : CQ is constant.

(II.) Take the centre as origin. Let P be (x', y'). Equation to TR is found thus.

[TR is the straight line passing through the intersection of TP and a straight line through the middle of CP and perpendicular to CP, and it makes an angle with x-axis whose tangent is opposite in sign to that of the inclination of TP with x-axis.]



Equation to TP is

$$xx'/a^2 + yy'/b^2 = 1 \quad \text{.....(i.).}$$

Equation to the straight line through the middle point of CP and perpendicular to CP is

$$(y - \tfrac{1}{2}y') = (-x'/y')(x - \tfrac{1}{2}x') \text{.....(ii.).}$$

Equation to any straight line through intersection of (i.) and (ii.) is

$$(xx'/a^2 + yy'/b^2 - 1) - \lambda \left\{(y - \tfrac{1}{2}y') + (x'/y')\,(x - \tfrac{1}{2}x')\right\} = 0.$$

This will represent TR if

$$\lambda x'/y' - x'/a^2 = (b^2x'/a^2y')\,(y'/b^2 - \lambda) \quad \text{or} \quad \lambda = 2y'/(a^2 + b^2).$$

Therefore $CR = \pm\, ab\,[x'^2 + y'^2 - (a^2 + b^2)]/[(a^2 - b^2)b']$,

with usual notation. But $x'^2 + y'^2 - (a^2 + b^2) = -b'^2$, or absolute value of $CR = (ab \,.\, b')/(a^2 - b^2)$. Therefore $CR : CQ = ab : (a^2 - b^2)$.

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15163. (Rev. T. ROACH, M.A.)—If A, B, C be the angles of a triangle, show that $\begin{vmatrix} \sin A & \cos A & \cos(C-A) \\ \sin B & \cos B & \cos(B-C) \\ \sin C & \cos C & 1 \end{vmatrix} = 0.$

Solution by R. TUCKER, M.A., F. W. REEVES, B.A., *and* C. M. ROSS.

Multiply the constituents of the first and second columns by sin C and cos C respectively, and add the products to make a new first column. The result will be the third column. Hence, &c. N.B.—A, B, C may be any angles.

9515. (R. HOLMES, B.A.)—If P_n is LEGENDRE'S coefficient of order n, and J_0 is the BESSEL'S function of order zero, show that

$$P_0(\mu) - rP_1(\mu) + \frac{r^2}{2!}P_2(\mu) - \frac{r^3}{3!}P_3(\mu) + \ldots + (-1)^n \frac{r^n}{n!}P_n(\mu) + \ldots = e^{-\mu r} J_0\{r(1-\mu^2)^{\frac{1}{2}}\}.$$

Solution by G. H. HARDY, M.A., *and* R. F. WHITEHEAD.

If $\nu = (1-\mu^2)^{\frac{1}{2}}$, $\quad P_n(\mu) = \int_0^\pi (\mu + i\nu\cos\phi)^n \, d\phi$,

$$\sum_0^\infty \frac{(-)^n r^n}{n!} P_n(\mu) = \int_0^\pi d\phi \left[\sum_0^\infty \frac{(-1)^n r^n (\mu + i\nu\cos\phi)^n}{n!}\right] = \int_0^\pi e^{-r(\mu + i\nu\cos\phi)} d\phi$$
$$= e^{-r\mu} \int_0^\pi \cos(r\nu\cos\phi)\, d\phi = e^{-r\mu} J_0(r\nu).$$

15329. (Professor SANJÁNA, M.A.)—In a nodal cube, N being the node, a variable chord ABC is so drawn that the tangents at A and B meet on the curve at D. Prove that N (ABCD) is an involution of which the nodal tangents are double rays. (In this case the satellite of ABC touches the curve.)

Solution by Professor NANSON.

Any cubic through the reference point is $ux^2 + vy^2 + wz^2 = kxyz$, where u, v, w are the tangents at those points. If now A, B, N are the reference points, we have $w \equiv 0$, because N is a node; and we have $k = 0$, because the tangents at A, B move on the curve. Thus the equation reduces to $(ay + a'z)\,x^2 + (bx + b'z)\,y^2 = 0$.

Hence the tangents at the node are $a'x^2 + b'y^2 = 0$, NA, NB are $xy = 0$, NC is $ax + by = 0$, and ND is $ab'y - a'bx = 0$; whence it is obvious that NA, NB, as also NC, ND, are harmonic conjugates with respect to the nodal tangents.

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8474. (Professor GILLILAND, M.A.)—A closed conical vessel whose height is twice the diameter of the base is just filled with liquid and suspended from a point in the rim of its base. Prove that, if the weight of the vessel be neglected, the vertical component of the pressure on its curved surface will be $\frac{1}{8}$ of the weight of the liquid. [N.B.—In the original printing of the Question there is a misprint in the fraction given.]

Solution by Professor SANJÁNA, M.A., *and* A. M. NESBITT, M.A.

If α be the semi-vertical angle, r radius of base, ρ specific gravity, the weight of the liquid $= \frac{1}{3}g\rho\pi r^3 \cot\alpha$. Project the curved surface on the horizontal plane through the point of suspension, and we see that the vertical pressure on it = weight of liquid + vertical pressure on base. The latter being $g\rho\pi r^2 . r\cos\frac{1}{4}\pi . \sin\frac{1}{4}\pi$, we get

$$\text{required pressure} = g\rho\pi r^3(\tfrac{1}{3}\cot\alpha + \tfrac{1}{2}) = 3W\tan\alpha(\tfrac{1}{3}\cot\alpha + \tfrac{1}{2})$$
$$= W(1 + \tfrac{3}{2}\tan\alpha) = W(1 + \tfrac{3}{2}.\tfrac{1}{4}) = \tfrac{1}{8}\text{ (weight of liquid)}.$$

15250. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—State conditions under which one, or both, of $N_1 = (a^n-1)^3-1$, $N_2 = (a^n+1)^3+1$ are completely resolvable into *prime* factors, when $a^n > 3000$. *Ex.*—Show that both are possible when $a^n = 2^{70}, 3^{15}, 5^{10}, 10^7$.

Solution by the PROPOSER.

$$\frac{N_1}{a^n-2} = \frac{(a^n-1)^3-1}{(a^n-1)-1} = \frac{a^{3n}+1}{a^n+1}, \quad \frac{N_2}{a^n+2} = \frac{(a^n+1)^3+1}{(a^n+1)+1} = \frac{a^{3n}-1}{a^n-1};$$

therefore N_1, N_2 are resolvable into prime factors if $(a^{3n}\pm1)$ are so resolvable. Some of the cases are :—

(i.) The existing large factor-tables extend only to nine million. Hence all such numbers are resolvable when $a^n \not> 3000$.

(ii.) $(a^{3n}\pm1)\div(a^n\pm1)$ always contains 3, if $(a^n\pm1)$ does; hence N_1 or N_2 will be resolvable if $a^n \not> \sqrt{27}$ million, *i.e.*, $\not> 5291$.

(iii.) Since $N = \{(3x^2)^3+1\}+(3x^2+1)$ is always *algebraically* resolvable into two factors, L, M, where L or $M = (3x^2\mp3x+1)$, it follows that N_1 will be resolvable if $a^n = 3a^2$, and N_2 will be so resolvable if $a^n+1 = 3a^2$, and they will be *completely* resolvable if the factors L, M $\not> 9$ million, *i.e.*, if $a^n \not> 81$ billion.

(iv.) With the help of *special divisors*, the complete factorisation can be carried much further. The PROPOSER has compiled a table of solutions of the congruence $(x^3\pm1)\div(x\pm1) \equiv 0 \pmod p$ for all primes $\not> 10{,}000$. This increases the limit of factorisability from 9 to 100 million. (This table will, it is hoped, be published shortly.)

(v.) The examples quoted depend on the following factorisations, due to the authors stated :—$(2^{210}\pm1)$ by MM. AURIFEUILLE and LELASSEUR; $(3^{45}\pm1)$ by Mr. CHAS. E. BICKMORE; $(5^{30}-1)$ by Herr REUSCHLE; $(5^{30}+1)$ by the PROPOSER; $(10^{21}\pm1)$ by Herr LOOFF.

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15368. (Professor NANSON.)—If lines drawn through the vertices of a triangle of area Δ divide the angles into the segments α, α'; β, β'; γ, γ', and form a triangle of area Δ', then

$$\frac{\Delta'}{\Delta} = \frac{(\sin\alpha\sin\beta\sin\gamma - \sin\alpha'\sin\beta'\sin\gamma')^2}{\sin(\beta+\gamma')\sin(\gamma+\alpha')\sin(\alpha+\beta')\sin(\alpha+\alpha')\sin(\beta+\beta')\sin(\gamma+\gamma')}.$$

Solutions (I.) *by* R. TUCKER, M.A., Professor SANJÁNA, M.A., *and* C. M. ROSS; (II.) *by* J. H. TAYLOR, M.A.

(I.) Referring to *Reprint* (New Series, Vol. III., p. 74), we have

$$RP(\sin\alpha+\beta')\sin(\beta+\gamma') = 2R(\Pi\sin\alpha - \Pi\sin\alpha');$$

hence
$$\frac{\Delta'}{\Delta} = \frac{PQ.QR.RP}{(4R')}\frac{4R}{abc}$$
$$= \frac{R(\Pi\sin\alpha - \Pi\sin\alpha')^3}{R'\Pi(\sin\alpha+\beta')^2\Pi\sin(\alpha+\alpha')} = \frac{(\Pi\sin\alpha - \Pi\sin\alpha')^2}{\Pi\sin(\alpha+\beta')\Pi\sin(\alpha+\alpha')}.$$

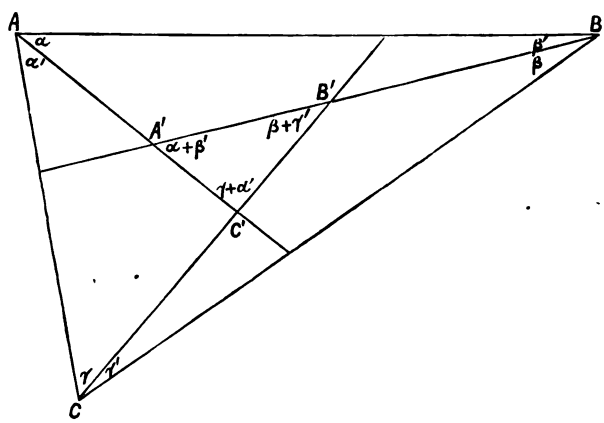
(II.)
$$AA' = \frac{c\sin\beta'}{\sin(\alpha+\beta')}, \qquad AC' = \frac{b\sin\gamma}{\sin(\gamma+\alpha')};$$

therefore
$$A'C' = \frac{b\sin\gamma\sin(\alpha+\beta') - c\sin\beta'\sin(\gamma+\alpha')}{\sin(\gamma+\alpha')\sin(\alpha+\beta')}$$

and
$$A'B' = \frac{c\sin\alpha\sin(\beta+\gamma') - a\sin\gamma'\sin(\alpha+\beta')}{\sin(\alpha+\beta')\sin(\beta+\gamma')}.$$

$$\Delta = \frac{bc\sin(\alpha+\alpha')}{2}, \qquad \Delta' = A'B'.A'C'\sin C'A'B';$$

$$\therefore \frac{\Delta'}{\Delta} = \frac{\{\sin(\beta+\beta')\sin\gamma\sin(\alpha+\beta') - \sin(\gamma+\gamma')\sin\beta'\sin(\gamma+\alpha')\}}{\times\{\sin(\gamma+\gamma')\sin\alpha\sin(\beta+\gamma') - \sin(\alpha+\alpha')\sin\gamma'\sin(\alpha+\beta')\}}$$
$$= \frac{\{\sin\alpha\sin\beta\sin\gamma - \sin\alpha'\sin\beta'\sin\gamma'\}^2}{\sin(\alpha+\alpha')\sin(\beta+\beta')\sin(\gamma+\gamma')\sin(\alpha+\beta')\sin(\beta+\gamma')\sin(\gamma+\alpha')},$$



since the factors of the numerator are equal and are easily reduced, *e.g.*,

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$$\begin{aligned}
&\sin(\beta+\beta')\sin\gamma\sin(\alpha+\beta')-\sin(\gamma+\gamma')\sin\beta'\sin(\gamma+\alpha')\\
&\quad=\sin\gamma\{\sin\alpha\sin\beta(1-\sin^2\beta')+\sin\alpha\cos\beta\sin\beta'\cos\beta'\\
&\qquad\qquad+\cos\alpha\sin\beta\sin\beta'\cos\beta'+\cos\alpha\cos\beta\sin^2\beta'\}\\
&\qquad-\sin\beta'\{\sin^2\gamma\cos\alpha'\cos\gamma'+\sin\gamma\cos\gamma\cos\alpha'\sin\gamma'\\
&\qquad\qquad+\sin\gamma\cos\gamma\sin\alpha'\cos\gamma'+\sin\alpha'\sin\gamma'(1-\sin^2\gamma)\}\\
&\quad=\sin\gamma\{\sin^2\beta'\cos(\alpha+\beta)+\sin\beta'\cos\beta'\sin(\alpha+\beta)\}+\sin\alpha\sin\beta\sin\gamma\\
&\qquad-\sin\beta'\{\sin^2\gamma\cos(\alpha'+\gamma')+\sin\gamma\cos\gamma\sin(\alpha'+\gamma')\}-\sin\alpha'\sin\beta'\sin\gamma'\\
&\quad=\sin\beta'\sin\gamma\{\sin(\alpha+\beta+\beta')-\sin(\gamma+\alpha'+\gamma')\}\\
&\qquad\qquad+\sin\alpha\sin\beta\sin\gamma-\sin\alpha'\sin\beta'\sin\gamma\\
&\quad=\sin\alpha\sin\beta\sin\gamma-\sin\alpha'\sin\beta'\sin\gamma',
\end{aligned}$$

since $\alpha+\beta+\beta'$ and $\gamma+\alpha'+\gamma'$ are supplementary.

9143. (A. E. Thomas.)—A circle described through the origin, and touching the curve $(x^2+y^2-2ax)^2(x^2+y^2-5ax)=4a^3y^3$ at a point Q, cuts the parabola $y^2=4ax$ in three points, the normals at which to the parabola meet in a point W. Find the locus of W as Q varies.

Solution by Frances E. Cave.

Let (ξ, η) be W, and let the equation of the circle be

$$x^2+y^2+2gx+2fy=0.$$

Where this meets $y^2=4ax$ we have $y^3/8a^2+y(2+g/a)+4f=0$,

and the feet of normals from W are given by $y^3/8a^2+y(1-\xi/2a)-\eta=0$. Therefore $4f=-\eta$, $a+\frac{1}{2}\xi+g=0$. Where the circle meets the given curve we therefore have

$$\{x\xi+\tfrac{1}{2}y\eta\}^2\{x(\xi-3a)+\tfrac{1}{2}y\eta\}-4a^3y^3=0,$$

a cubic in x/y. Since the circle touches the curve, two of the roots of this cubic are equal, and hence $y^3-8x(x-3a)^2=0$ is the equation of the locus of W.

15177. (R. Tucker, M.A.)—In the "ambiguous" case AC = AC′, AL, AL′ are the respective symmedians (through A) of the triangles ABC, ABC′, cutting BC in L, L′ respectively. Find (i.) LL′, (ii.) the value of BC when AC′ is the symmedian of ABC.

Solution by T. Savage, B.A., *and the* Proposer.

$$\mathrm{BL}=ac^2/(b^2+c^2),\qquad \mathrm{CL}=ab^2/(b^2+c)$$

[Nixon's *Elementary Plane Trigonometry*, chapter xiv.]; therefore

(i.) $\mathrm{LL'}=\mathrm{BL}-\mathrm{BL'}=(a-a')/[c^2/(b^2+c^2)]$,

or, denoting BC′ by a', $\qquad=2b\cos\mathrm{C}\,[c^2/(b^2+c^2)]$.

(ii.) $\mathrm{CC'}=ab^2/(b^2+c^2)$. But $\mathrm{CC'}=2b\cos\mathrm{C}$; therefore

$$\mathrm{BC}=2(b^2+c^2)\cos\mathrm{C}/b.$$

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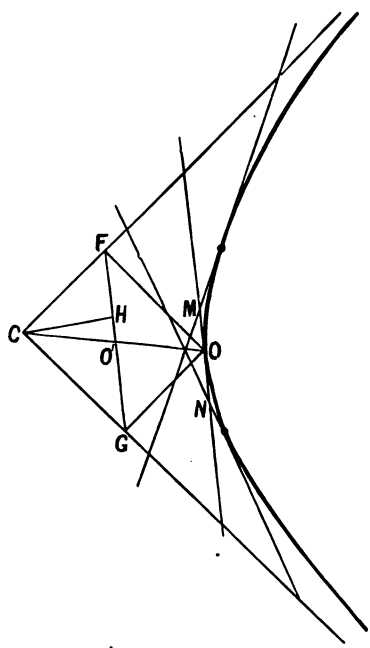
15121. (R. Knowles.)—Two tangents are drawn to meet a rectangular hyperbola centre C; a third tangent meets these in M, N; O is the mid-point of MN; F, G are the projections of O on the asymptotes; O′ is the mid-point of FG. Prove that (1) the points O, O′, C are collinear; (2) if H is the projection of C on FG, the angle HCO is bisected by the axes.

Solutions (I.) *by* A. M. Nesbitt, M.A.; (II.) *by* A. W. Poole.

(I.) (1) OGCF is a rectangle; therefore the middle point of FG is on the other diagonal OC.

(2) Let (a, b) be co-ordinates of O. Then the equation to FG is $x/a + y/b = 1$; therefore the equation to CH is $ax = by$, and that to CO is $bx = ay$. Two lines bisecting the angles between CH and CO are therefore $x \pm y = 0$; and these are the axes of any hyperbola which has the axes of co-ordinates for asymptotes.



The existence of the hyperbola and its three tangents is, in fact, quite unnecessary; and even (2) can be easily solved by elementary geometry. For $\angle GCH = \angle OCF$; therefore $\angle HCO$ is bisected by the line which bisects FCG.

(II.) (1) In the Figure CGOF is a rectangle; therefore its diagonals bisect each other; therefore O, O′, C are collinear.

(2) $\angle FCH = \frac{1}{2}\pi - \angle CFH$
$= \angle FGC = \angle O'CG$;
therefore the exterior angle HCO is bisected by the axes.

10544. (H. W. Segar.)—Given the lengths of four lines that may be drawn from a point to the angular points of a rhombus, describe the rhombus.

Solutions (I.) *by* W. H. Blythe, M.A.; (II.) *by* D. Biddle.

(I.) Let a, b, c, d be the distances of a point from the angular points of a rhombus, the semi-diagonals of which are l and m. Since the diagonals

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bisect each other at right angles, take these as rectangular axes, and let x, y be the co-ordinates of the point referred to these axes. Then

$$(x-m)^2+y^2=a^2, \quad x^2+(y-l)^2=b^2 \ldots\ldots\ldots\ldots\ldots\ldots (1,2),$$
$$(x+m)^2+y^2=c^2, \quad x^2+(y+l)^2=d^2 \ldots\ldots\ldots\ldots\ldots\ldots (3,4).$$

If we take $x^2+y^2=r^2$ and eliminate x, y, l, m, we find

$$8r^2(a^2+c^2-2r^2)(b^2+d^2-2r^2)=(b^2+d^2-2r^2)(c^2-a^2)^2+(a^2+c^2-2r^2)(d^2-b^2)^2.$$

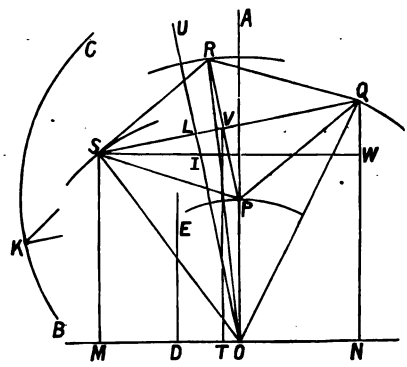
Now suppose this cubic for r^2 solved, for a, b, c, d are known; then

$$2l^2=b^2+d^2-2r^2 \text{ and } 2m^2=a^2+c^2-2r^2;$$

so that l and m are completely determined, and the rhombus can be constructed.

(II.) Let O be the centre of concentric circles having the given lengths as radii. Let P be the nearest angular point of the rhombus, and through it draw OA. Let

APR $=\theta$, OP $=p$,
OQ $=q$, OR $=r$,
OS $=s$, QS $=2y$,
PR $=2z$, PQ $=x$.



Analytically. — Draw OU parallel to PR, cutting QS at right angles in L. Then OL $=z+p\cos\theta$, whilst QL $=y+p\sin\theta$, and SL $=y-p\sin\theta$.

It follows that $4py\sin\theta=q^2-s^2$. Similarly $4z^2+4zp\cos\theta=r^2-p^2$.

We also have $(p\cos\theta+z)^2+(y-p\sin\theta)^2=s^2$, whence

$$y^2-z^2=\tfrac{1}{2}(q^2+s^2-r^2-p^2).$$

If therefore we describe the arc BC with centre P and radius

$$\left\{2(q^2+s^2-r^2-p^2)\right\}^{\frac{1}{2}},$$

we know that, if PK be drawn at right angles to PR, we have RK $=2y$; and that, if we take OD $=\frac{1}{2}(q^2-s^2)$ at right angles to OA, and draw DE parallel to OA, it will meet PR produced at a distance $2py$ from P. But, in order to find the exact values of x, y, and z, we must apparently have recourse to a cubic equation. Let

$$q^2-s^2=4a, \quad r^2-p^2=4b, \quad 2(q^2+s^2-r^2-p^2)=4c.$$

Then we have

$$4py\sin\theta=4a, \quad 4z^2+4pz\cos\theta=4b, \quad 4y^2-4z^2=4c \ldots\ldots (1,2,3),$$

whence we obtain the following cubic in y^2:—

$$y^6-\left\{2(b+c)+p^2\right\}y^4+\left\{a^2+(b+c)^2+cp^2\right\}y^2-a^2c=0 \ldots\ldots\ldots (4).$$

Let $\quad 2(b+c)+p^2=3f, \quad a^2+(b+c)^2+cp^2=g, \quad a^2c=h.$

Then, taking $u+f=y^2$, we have

$$u^3-(3f^2-g)u-(2f^3-fg+h)=0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

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which can in certain cases be solved by CARDAN's method, namely, when $\frac{1}{2}\{(2f^3-fg+h)^2\} > \frac{1}{27}\{(3f^2-g)^3\}$, and this obtains when

$$27h^2+108f^3h+4g^3 > 9f^2g^2+54fgh.$$

Thus it enables us to solve the case in which $p=1$, $q=3$, $r=4$, $s=2$, giving x, the side of the rhombus, as $2\frac{3}{8}$ nearly, y being about 1·25, and z about 1·85. In this case the whole of the rhombus lies to one side of OA, and PR is the long diagonal, OPR being little greater than a right angle.

15323. (Rev. T. ROACH, M.A.)—Prove that the series

$$\frac{1}{1.2.6.7}-\frac{1}{3.4.8.9}+\frac{1}{5.6.10.11}-\ldots=\frac{1}{60}\left(\pi-\frac{149}{60}\right).$$

Solutions (I.) *by* Rev. T. WIGGINS, B.A.; (II.) *by* Professor SANJÁNA, M.A.; (III.) *by the* PROPOSER.

(I.) Expression is

$$\Sigma_{n=\infty}^{n=1}\frac{(-1)^{n+1}}{(2n-1)(2n)(2n+4)(2n+5)}$$

$$=\Sigma_{n=\infty}^{n=1}(-1)^{n+1}\left(\frac{1}{30\,(2n-1)}-\frac{1}{20\,.\,2n}+\frac{1}{20\,(2n+4)}-\frac{1}{30\,(2n+5)}\right)$$

$$=\frac{1}{30}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\ldots\right)-\frac{1}{20}\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\ldots\right)$$
$$+\frac{1}{20}\left(\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\ldots\right)-\frac{1}{30}\left(\frac{1}{7}-\frac{1}{9}+\frac{1}{11}-\ldots\right)$$

$$=(x=1)\,\frac{1}{30}\left(\frac{x}{1}-\frac{x^3}{3}+\ldots\right)-\frac{1}{20}\left(\frac{x^2}{2}-\frac{x^4}{4}+\ldots\right)$$
$$+\frac{1}{20}\left(\frac{x^6}{6}-\frac{x^8}{8}+\ldots\right)-\frac{1}{30}\left(\frac{x^7}{7}-\frac{x^9}{9}+\ldots\right)$$

$$=\int_0^1\left\{\frac{1}{30}(1-x^2+x^4-\ldots)-\frac{1}{20}(x-x^3+x^5-\ldots)+\frac{1}{20}(x^5-x^7+\ldots)\right.$$
$$\left.-\frac{1}{30}(x^6-x^8+\ldots)\right\}dx$$

$$=\int_0^1\left\{\frac{1}{30}\frac{1}{1+x^2}-\frac{1}{20}\frac{x}{1+x^2}+\frac{1}{20}\frac{x^5}{1+x^2}-\frac{1}{30}\frac{x^6}{1+x^2}\right\}dx$$

$$=\int_0^1\left\{\frac{1}{30}\frac{1-x^6}{1+x^2}-\frac{x}{20}\frac{1-x^4}{1+x^2}\right\}dx$$

$$=\int_0^1\frac{1}{30}\frac{1-x^6}{1+x^2}dx-\frac{1}{20}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)_0^1$$

$$=\frac{1}{80}-\frac{1}{40}+\int_0^1\frac{1}{30}\left(\frac{2-(1+x^2)+(x^2-x^6)}{(1+x^2)}\right)dx$$

$$=\frac{1}{80}-\frac{1}{40}+\frac{1}{30}\left(2\tan^{-1}1-1+\frac{1}{3}-\frac{1}{5}\right)$$

$$=\frac{2}{30}\frac{\pi}{4}+\left(\frac{1}{80}-\frac{1}{40}-\frac{1}{30}+\frac{1}{90}-\frac{1}{150}\right)=\frac{1}{60}\left(\pi-\frac{149}{60}\right).$$

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(II.) The given series

$$= \frac{1}{40}\frac{42-2}{1.2.6.7} - \frac{1}{60}\frac{72-12}{3.4.8.9} + \frac{1}{80}\frac{110-30}{5.6.10.11} \cdots$$

$$- \frac{1}{40}\left(\frac{1}{1} - \frac{1}{2} - \frac{1}{6} + \frac{1}{7}\right) - \frac{1}{60}\left(\frac{1}{3} - \frac{1}{4} - \frac{1}{8} + \frac{1}{9}\right)$$
$$+ \frac{1}{80}\left(\frac{1}{5} - \frac{1}{6} - \frac{1}{10} + \frac{1}{11}\right) \cdots$$

$$= \frac{1}{40}\left(\frac{8}{1.7} - \frac{8}{2.6}\right) - \frac{1}{60}\left(\frac{12}{3.9} - \frac{12}{4.8}\right) + \frac{1}{80}\left(\frac{16}{5.11} - \frac{16}{6.10}\right) \cdots$$

$$= \frac{1}{5}\left(\frac{1}{1.7} - \frac{1}{3.9} + \frac{1}{5.11} - \frac{1}{7.13} + \frac{1}{9.15} - \frac{1}{11.17} \cdots\right)$$
$$- \frac{1}{5}\left(\frac{1}{2.6} - \frac{1}{4.8} + \frac{1}{6.10} - \frac{1}{8.12} + \frac{1}{10.14} - \frac{1}{12.16} \cdots\right)$$

$$= \frac{1}{30}\left(\frac{1}{1} - \frac{1}{7} - \frac{1}{3} + \frac{1}{9} + \frac{1}{5} - \frac{1}{11} - \frac{1}{7} + \frac{1}{13} + \frac{1}{9} - \frac{1}{15} - \frac{1}{11} \cdots\right)$$
$$- \frac{1}{20}\left(\frac{1}{2} - \frac{1}{6} - \frac{1}{4} + \frac{1}{8} + \frac{1}{6} - \frac{1}{10} - \frac{1}{8} + \frac{1}{12} + \frac{1}{10} - \frac{1}{14} - \frac{1}{12} \cdots\right)$$

$$= \frac{1}{30}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} \cdots\right) - \frac{1}{20}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{1}{30}\left(\frac{\pi}{2} - \frac{1}{1} + \frac{1}{3} - \frac{1}{5}\right) - \frac{1}{80} = \frac{\pi}{60} - \frac{149}{3600} = \frac{1}{60}\left(\pi - \frac{149}{60}\right).$$

(III.)

$$\frac{1}{1.2.6.7} + \frac{1}{5.6.10.11} + \cdots$$

$$= \frac{1}{60}\left(\frac{2}{1} - \frac{3}{2} + \frac{3}{6} - \frac{2}{7} + \frac{2}{5} - \frac{3}{6} + \frac{3}{10} - \frac{2}{11} + \frac{2}{9} - \frac{3}{10} + \frac{3}{14} - \frac{2}{15} + \cdots\right)$$

$$= \frac{1}{60}\left(-\frac{3}{2} + \frac{2}{1} + \frac{2}{5} - \frac{2}{7} + \frac{2}{9} - \frac{2}{11} + \frac{2}{13} - \frac{2}{15} \cdots\right)$$

$$= \frac{1}{60}\left\{-\frac{3}{2} + \frac{2}{3} + 2\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)\right\}$$

$$= \frac{1}{60}\left(-\frac{5}{6} + \frac{\pi}{2}\right),$$

$$\frac{1}{3.4.8.9} + \frac{1}{7.8.12.13} + \cdots$$

$$= \frac{1}{60}\left(\frac{2}{3} - \frac{3}{4} + \frac{3}{8} - \frac{2}{9} + \frac{2}{7} - \frac{3}{8} + \frac{3}{12} - \frac{2}{13} + \frac{2}{11} - \frac{3}{12} + \frac{3}{16} - \frac{2}{17} + \cdots\right)$$

$$= \frac{1}{60}\left\{-\frac{3}{4} + 2 + \frac{2}{5} - 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} \cdots\right)\right\}$$

$$= \frac{1}{60}\left(\frac{33}{20} - 2\frac{\pi}{4}\right);$$

therefore $\frac{1}{1.2.6.7} - \frac{1}{3.4.8.9} + \frac{1}{5.6.10.11} - \cdots$

$$= \frac{1}{60}\left(-\frac{5}{6} + \frac{\pi}{2} - \frac{33}{20} + \frac{\pi}{2}\right) = \frac{1}{60}\left(\pi - \frac{149}{60}\right).$$

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15288. (Rev. Prebendary W. A. WHITWORTH, M.A.)—Show that there are *only* five triangles whose sides are integers and whose area is represented by the same integer as the sum of the sides.

Solution by D. BIDDLE.

It is easier to find five triangles fulfilling the conditions of the Question than to prove that they are the only ones. Thus, the following are readily found, consisting of two which are right-angled and of three which are the remainders left on subtracting one right-angled triangle from another having a side in common:

(i.) 5, 12, 13; (ii.) 6, 8, 10; (iii.) 7, 15, 20 = (12, 16, 20)−(12, 9, 15);
(iv.) 9, 10, 17 = (8, 15, 17)−(8, 6, 10);
(v.) 6, 25, 29 = (20, 21, 29)−(20, 15, 25).

Taking first the case of right-angled triangles, let m, n be the two sides, and $(m^2+n^2)^{\frac{1}{2}}$ the hypotenuse. Then, by the conditions,

$$\tfrac{1}{2}mn = m+n+(m^2+n^2)^{\frac{1}{2}} \quad \ldots\ldots (1);$$

whence

$$mn-4m-4n+8 = 0 \text{ and } n = (4m-8)/(m-4) = 4+8/(m-4) \quad (2, 3),$$

This is satisfied by taking $m = 5$, 6, 8, or 12; whence $n = 12$, 8, 6, or 5; and $(m^2+n^2)^{\frac{1}{2}} = 13$, 10, 10, or 13. But the triangles thus formed are two only, (i.), (ii.).

In order to test other triangles, let $a > b > c$, and let a', b', c' be the respective perpendiculars, whereof $a' < b' < c'$. Then we have

$$a'a = b'b = c'c = 2(a+b+c) \text{ and } a'a+b'b+c'c = 6(a+b+c) \quad (4, 5);$$

whence

$$(c'-6)c+(b'-6)b+(a'-6)a = 0 \quad \ldots\ldots (6),$$

where it is clear that one of the three terms is negative, and we have

$$\begin{aligned}(6-a')a &= (b'-6)b+(c'-6)c \quad \ldots\ldots (7)\\ &= 4(a+b+c)-6(b+c), \text{ by (4)},\\ &= 4a-2(b+c),\end{aligned}$$

in which we know the right side to be positive. Therefore

$$a' < 6 \quad \ldots\ldots (8).$$

Moreover, since the sum of the sides of any triangle cannot be less than twice its longest side, we have, by (4), $a'a > 4a$; whence (8) becomes

$$4 < a' < 6 \quad \ldots\ldots (9).$$

All this holds good in the triangles (iii.), (iv.), (v.). But it is not quite clear that, as in their case, the perpendicular to one of the sides must necessarily be integral, as well as the sides themselves of the required triangle. For, if so, it follows, from (4), that of $2(a+b)/c$, $2(a+c)/b$, $2(b+c)/a$, one also must be integral, the first-mentioned being the most likely, and such obtains in (iii.), (iv.), (v.). Nor is it much clearer that the required triangle must needs be right-angled or obtuse; and to prevent ambiguity we will consider it always as the sum of two

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right-angled triangles having a' as a common side, for the most part non-integral. Let the foot of a' divide a into a_1, a_2, such that

$$a' = (b^2-a_1^2)^{\frac{1}{2}} = (c^2-a_2^2)^{\frac{1}{2}}.$$

Then we have

$$b^2-c^2 = a_1^2-a_2^2 = (a_1+a_2)(a_1-a_2) = aa_1-aa_2 \ldots\ldots\ldots\ldots (10).$$

Consequently, a_1+a_2 is integral, $a(a_1+a_2)$ is integral, aa_1, aa_2 are integral, $a'(a_1+a_2)$ is integral; also $a'^2+a_1^2$, $a'^2+a_2^2$ are integral. Therefore a_1, a_2 must have fractions (if any) which are complementary to each other, and these must both result, on squaring, in fractions which are complementary to that of a'^2. Yet the denominator of the root-fraction must be a factor of a, just as that of a' is.

We thus arrive at the following law:—If two right-angled triangles can be found having integral sides one of which is common, and whose hypotenuses and the sum of the third sides have a common factor (f), the sum of the third sides having the factor f^2; if also the area of the triangle resulting from apposition of these two can be represented by f times the sum of its sides, then is this triangle one whose sides are severally f times the length of those belonging to a triangle fulfilling the conditions of the Question; and, clearly, the common side of the two component (integral) triangles lies between $4f$ and $6f$.

The sides of the two component triangles sought are fa', fa_1, fb and fa', fa_2, fc. In order to reduce the two to standard right-angled triangles with integral sides, divide the sets by q, r respectively, and let the sides of the standards concerned be μ_1, ν_1, β and μ_2, ν_2, γ. Then we have $fa' = q\mu_1 = r\mu_2$, $fa_1 = q\nu_1$, $fb = q\beta$, whilst $fa_2 = r\nu_2$, $fc = r\gamma$. It is easy to see that $\mu_1 : \mu_2 = r : q$; also that $q\beta$, $r\gamma$ have the common factor f, whilst $q\nu_1 + r\nu_2$ has the factor f^2, which is also the case with $\mu_2\nu_1 + \mu_1\nu_2$. But we have

$$\tfrac{1}{2}a'(q\nu_1/f + r\nu_2/f) = (q\beta + r\gamma + q\nu_1 + r\nu_2)/f \ldots\ldots\ldots\ldots (11);$$

whence

$$(q\nu_1 + r\nu_2)/2f = (\beta+\nu_1)/\mu_1 + (\gamma+\nu_2)/\mu_2 \ldots\ldots\ldots\ldots (12)$$

or

$$a = 2(\beta+\nu_1)/\mu_1 + 2(\gamma+\nu_2)/\mu_2 \ldots\ldots\ldots\ldots (13),$$

where the two terms to the right are severally integral. But, since $\mu_1^2 = \beta^2-\nu_1^2$, and $\mu_2^2 = \gamma^2-\nu_2^2$, we can render (13) as

$$a = 2\mu_1/(\beta-\nu_1) + 2\mu_2/(\gamma-\nu_2) \ldots\ldots\ldots\ldots (14).$$

Again, $\mu_1 : a' = \beta : b$ and $\mu_2 : a' = \gamma : c$; whence $b = \beta a'/\mu_1$ and $c = \gamma a'/\mu_2$. Therefore, by (13), we have

$$a = 2(b+c)/a' + 2(\nu_1/\mu_1 + \nu_2/\mu_2);$$

whence, and from (4), it follows that

$$a/a' = \nu_1/\mu_1 + \nu_2/\mu_2 \ldots\ldots\ldots\ldots (15).$$

Thus

$$a' = \mu_1\mu_2 a/(\nu_1\mu_2 + \nu_2\mu_1) = 2 + 2(\beta\mu_2 + \gamma\mu_1)/(\nu_1\mu_2 + \nu_2\mu_1) \ldots\ldots\ldots (16),$$

and we have

$$1 < (\beta\mu_2 + \gamma\mu_1)/(\nu_1\mu_2 + \nu_2\nu_1) < 2 \ldots\ldots\ldots\ldots (17),$$

where the second term has a denominator which we already know to be a multiple of f^2, whilst the numerator can be seen, by (13), to be a multiple of f, and this factor is common to β and γ. The equivalent of (17) is $1 < (b+c)/a < 2$.

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The product of the two terms to the right in (14) will be found to be either identical with, or a factor of, $a'a$, according as μ_1, μ_2 are both the shortest sides of their standards, or not. In the cases we are now considering (such as iii., iv., v.), they are both of this character. Therefore

$$a'a = 4\mu_1\mu_2/[(\beta-\nu_1)(\gamma-\nu_2)] \quad \ldots\ldots (18).$$

Also
$$1/a' = a/2(a+b+c) = \mu_1/2(\beta+\nu_1) + \mu_2/2(\gamma+\nu_2)$$
$$= (\beta-\nu)_1/2\mu_1 + (\gamma-\nu_2)/2\mu_2 \quad \ldots\ldots (19).$$

Wherefore, from (14), (15), (19), we have

$$(\beta-\nu_1)(\gamma-\nu_2)(\mu_2\nu_1+\mu_1\nu_2) = [(\beta-\nu_1)\mu_2+(\gamma-\nu_2)\mu_1]^2 \quad \ldots\ldots (20);$$

or, since $\mu_1^2 = \beta^2-\nu_1^2$ and $\mu_2^2 = \gamma^2-\nu_2^2$, we have

$$\mu_2\nu_1+\mu_1\nu_2 = 2\beta\gamma+2\mu_1\mu_2-2\nu_1\nu_2 \quad \ldots\ldots\ldots\ldots (21);$$

whence
$$\mu_2 = \gamma(4\nu_1-3\mu_1)/(5\beta) \quad \ldots\ldots\ldots\ldots (22).$$

In (iii.), $\mu_1 = 3$, $\nu_1 = 4$, $\beta = 5$; therefore $\mu_2 = \gamma \cdot \frac{7}{25}$, giving $\gamma = 25$, $\mu_2 = 7$, $\nu_2 = 24$. In (iv.), $\mu_1 = 8$, $\nu_1 = 15$, $\beta = 17$; therefore $\mu_2 = \gamma \cdot \frac{44}{85}$ giving $\gamma = 85$, $\mu_2 = 36$, $\nu_2 = 77$. In (v.), $\mu_1 = 20$, $\nu_1 = 21$, $\beta = 29$; therefore $\mu_2 = \gamma \cdot \frac{84}{145}$, giving $\gamma = 145$, $\mu_2 = 24$, $\nu_2 = 143$. In all these, 5β and $4\nu_1-3\mu_1$ are prime to each other. But it is requisite that the values found allow of $2\mu_2/(\gamma-\nu_2)$ being integral, as well as $2\mu_1/(\beta-\nu_1)$, so that, by (14), a may be integral also. For, if we take $\mu_1 = 5$, $\nu_1 = 12$, $\beta = 13$, we obtain, by (22), $\mu_2 = 33$, $\nu_2 = 56$, $\gamma = 65$; but these values, although belonging to a right-angled triangle, do not satisfy the above proviso.

Now we know that $1/a'$ lies between $\frac{1}{3}$ and $\frac{1}{4}$. Therefore, by (19), we must find $(\beta-\nu_1)/2\mu_1 < \frac{1}{4}$, and $(\gamma-\nu_2)/2\mu_2 < \frac{1}{4} - (\beta-\nu_1)/2\mu_1$, whilst the sum of the two must be greater than $\frac{1}{3}$. In standard triangles the greatest value of $(\beta-\nu_1)/(2\mu_1)$, when it is the inverse of an integral and μ_1 is the shortest side, is $\frac{1}{7}$, that is, where $\mu_1 = 20$, $\nu_1 = 21$, $\beta = 29$, as in (v.). Generally, if $1/k$ represent such value,

$$(k^2-4) : (k^2+4) = \nu_1 : \beta \quad \ldots\ldots\ldots\ldots (23);$$

and
$$\mu_1 = 4k \text{ or a proportionate factor of it} \quad \ldots\ldots\ldots\ldots (24).$$

$\frac{1}{3}$ gives $\mu_1 : \nu_1 : \beta = 24 : 32 : 40 = 3 : 4 : 5$, as in (iii.).

$\frac{1}{4}$ gives $\mu_1 : \nu_1 : \beta = 28 : 45 : 53$; whence, by (22), we have $\mu_2 = 96$, $\nu_2 = 247\gamma = 265$, but here $2\mu_2/(\gamma-\nu_2)$ is not integral.

$\frac{1}{5}$ gives $\mu_1 : \nu_1 : \beta = 32 : 60 : 68 = 8 : 15 : 17$, as in (iv.).

$\frac{1}{6}$ gives $\mu_1 : \nu_1 : \beta = 36 : 77 : 85 = \mu_2 : \nu_2 : \gamma$ in (iv.).

$\frac{1}{10}$ gives $\mu_1 : \nu_1 : \beta = 40 : 96 : 104 = 5 : 12 : 13$, unproductive, as shown above.

$\frac{1}{11}$ gives $\mu_1 : \nu_1 : \beta = 44 : 117 : 125$; whence, by (22), we have $\mu_2 = 336$, $\nu_2 = 527$, $\gamma = 625$, but here again $2\mu_2/(\gamma-\nu_2)$ is not integral.

$\frac{1}{13}$ gives $\mu_1 : \nu_1 : \beta = 48 : 140 : 148 = 12 : 35 : 37$; whence we have $\mu_2 = 104$, $\nu_2 = 153$, $\gamma = 185$, but here also $2\mu_2/(\gamma-\nu_2)$ is not integral.

We need go no further, because $\frac{1}{13} + 1/k = \frac{1}{3}$ is impossible with $k > 12$, and all lesser values have been tried. Hence the five triangles (i.), ... (v.) are the only ones which fulfil the conditions of the Question.

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15376. (Professor NEUBERG.)—Trouver $\int \tan x \tan 2x \tan 3x\, dx$.

Solutions (I.) *by* GEORGE SCOTT, M.A., *and* Rev. T. WIGGINS, B.A.; (II.) *by* GEORGE A. CARSE, M.A., B.Sc. (Edin.).

(I.) $\tan x = \tan(3x-2x) = (\tan 3x - \tan 2x)/(1+\tan 3x \tan 2x)$.

Hence $\tan x \tan 2x \tan 3x = \tan 3x - \tan 2x - \tan x$;

therefore $\int \tan x \tan 2x \tan 3x\, dx = \int (\tan 3x - \tan 2x - \tan x)\, dx$

$$= -\tfrac{1}{3}\log\cos 3x + \tfrac{1}{2}\log\cos 2x + \log\cos x$$

$$= \log\{[(\cos 2x)^{\frac{1}{2}}\cos x]/(\cos 3x)^{\frac{1}{3}}\}.$$

In the same way $\int \tan lx \tan mx \tan nx\, dx$ can be solved if any one of the numbers l, m, n is equal to the sum of the other two.

(II.) $I = \int \tan x \tan 2x \tan 3x\, dx$. Put

$$\tan x = t, \quad d\tan x/dx = \sec^2 x = 1+\tan^2 x,$$

i.e., $$dt/dx = 1+t^2, \quad dx = dt/(1+t^2),$$

$$\tan 2x = 2\tan x/(1-\tan^2 x) = 2t/(1-t^2), \quad \tan 3x = (3t-t^3)/(1-3t^2),$$

$$I = \int t\,\frac{2t}{1-t^2}\,\frac{3t-t^3}{1-3t^2}\,\frac{dt}{1+t^2};$$

by the theory of partial fractions (see CHRYSTAL'S *Algebra*, Vol. I., chap. viii.), this may be written in the form

$$I = 2\int\left(\frac{at+b}{1-t^2}+\frac{ct+d}{1-3t^2}+\frac{et+f}{1+t^2}\right)dt,$$

and, solving for the constants a, b, c, d, e, f, we find $a=-\frac{1}{2}$, $b=0$, $c=1$, $d=0$, $e=-\frac{1}{2}$, $f=0$; therefore

$$I = 2\int\left(-\tfrac{1}{2}\frac{t\,dt}{1-t^2}+\frac{t\,dt}{1-3t^2}-\tfrac{1}{2}\frac{t\,dt}{1+t^2}\right)$$

$$= \tfrac{1}{2}\int\frac{d(1-t^2)}{1-t^2}-\tfrac{1}{3}\int\frac{d(1-3t^2)}{1-3t^2}-\tfrac{1}{2}\int\frac{d(1+t^2)}{1+t^2}$$

$$= \log(1-t^2)^{\frac{1}{2}}(1-3t^2)^{-\frac{1}{3}}(1+t^2)^{-\frac{1}{2}}$$

$$= \log(1-\tan^2 x)^{\frac{1}{2}}(1-3\tan^2 x)^{-\frac{1}{3}}(1+\tan^2 x)^{-\frac{1}{2}}.$$

14882. (C. TAYLOR, D.D., Master of St. John's College, Cambridge.)—A chord of a conic drawn from a fixed point on the curve varies as the projection of the parallel focal chord on the normal thereat.

Solutions (I.) *by* H. W. CURJEL, M.A.; (II.) *by* H. M. TAYLOR, M.A., F.R.S.

(I.) Let P be the fixed point and PR the chord normal at P and PP′ any other chord through P. Let KSK′, QSQ′ be the focal chords parallel to PR, PP′, and qq' be the projection of QQ′ on PR. Through X, a point in PR near to P, draw a straight line parallel to PP′, cutting the conic in pp'. Then

$Xp \cdot Xp' : XP \cdot XR = QS \cdot SQ' : KS \cdot SK'$

$= QQ'$ latus rectum : KK' latus rectum $= QQ' : KK'$.

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In the limit when X moves up to P,

$$Xp : XP = QQ' : qq' \quad \text{and} \quad Xp' : XR = PP' : PR;$$

therefore $PP' : PR = qq' : KK'$, *i.e.*, $PP' : qq' = PR : KK' =$ constant.

(II.) If the tangent and normal at the fixed point O be taken for axes, the equation of the conic may be written

$$ax^2 + 2bxy + cy^2 = 2y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation of a chord PQ drawn through the focus S (α, β) may be written

$$x - \alpha = \cot\theta\,(y - \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

If the ordinates of P, Q be y_1, y_2, then

$$(y_1 - y_2)^2 = 4\,(A\cot^2\theta - 2B\cot\theta + C)/(a\cot^2\theta + 2b\cot\theta + c)^2,$$

where $A = (b^2 - ac)\,\beta^2 + 2a\beta$, $B = (b^2 - ac)\,\alpha\beta + a\alpha - b\beta$,

$$C = (b^2 - ac)\,\alpha^2 - 2b\alpha + 1.$$

The conditions that the point S is a focus of the conic (1) are $A = C$, $B = 0$; in this case

$$(y_1 - y_2)^2 = 4A\,\mathrm{cosec}^2\theta/(a\cot^2\theta + 2b\cot\theta + c)^2 = A \,.\, OR^2,$$

where OR is the chord parallel to PQ.

9774. (E. Borel.)—On considère un pentagone quelconque. On trace les cercles qui passent chacun par les extrémités d'un côté et par le point de rencontre des deux côtés adjacents. Démontrer que les cercles contigus se coupent deux à deux en cinq points (autres que les sommets du pentagone) qui se trouvent sur une même circonférence.

Note by W. H. Salmon, B.A., B.Sc.

We may prove not only this theorem, but also an extension of it, by Miquel's theorem. This states that, if we are given five lines, the foci of the five parabolas which have four of the given lines as tangents lie on a circle. Now the circum-circles of the four triangles formed by any four straight lines meet in a point, viz., the focus of the inscribed parabola. Hence we may generalize the above theorem and say that the circum-circles of the ten triangles formed by any five straight lines intersect in five points (other than the intersections of the five lines) which lie on a circle.

6481. (C. W. Merrifield, F.R.S.)—Obtain as a determinant, in a symmetrical form, the Cartesian equation of a right line passing among any n given points (x_1, y_1), ..., (x_n, y_n), so that the sum of the squares of their distances from it shall be a minimum.

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Solution by Professor NANSON.

Let the line be $x\cos\alpha + y\sin\alpha - p = 0$ .....(1).

Then for $\Sigma(x_r\cos\alpha + y_r\sin\alpha - p)^2$ to be a minimum we must have, differentiating with respect to p and α,

$$\bar{x}\cos\alpha + \bar{y}\sin\alpha - p = 0 \quad \text{.....(2)},$$

$$F(\cos^2\alpha - \sin^2\alpha) + (A-B)\cos\alpha\sin\alpha + p(\bar{x}\sin\alpha - \bar{y}\cos\alpha) = 0 \quad \text{... (3)},$$

where $n\bar{x} = \Sigma x_r$, $n\bar{y} = \Sigma y_r$, $A = \Sigma y_r^2$, $B = \Sigma x_r^2$, $F = \Sigma x_r y_r$;

and, eliminating p, α from (1), (2), (3), we get for the required line

$$\begin{vmatrix} 2F & A-B & \bar{y} & x & x \\ A-B & -2F & -\bar{x} & y & y \\ y & -\bar{x} & 0 & 1 & 1 \\ \bar{x} & \bar{y} & 1 & & \\ x & y & 1 & & \end{vmatrix} = 0.$$

9699. (Professor BEYENS.)—Si le rapport à l'hemisphère du segment *d'une* base de la sphère est m/n, le rapport au rayon de l'hauteur du segment *à deux* bases qui resultera est égal à $2\sin\frac{1}{3}[\sin^{-1}(n-m)/n]$.

Solution by Professor G. B. M. ZERR.

Let x = altitude of segment of one base; then $\frac{1}{3}\pi x^2(3r-x)$ = its volume and $\frac{2}{3}\pi r^3$ = volume of hemisphere; then $2r^3/[x^2(3r-x)] = n/m$

and $$\frac{2r^3 - x^2(3r-x)}{2r^3} = \frac{3r^2(r-x) - (r-x)^3}{2r^3} = \frac{n-m}{n}.$$

Now let $\theta = \sin^{-1}(r-x)/2r$ or $(r-x)/r = 2\sin\theta$;

then $3\sin\theta - 4\sin^3\theta = \sin 3\theta = (n-m)/n$;

therefore $\theta = \frac{1}{3}[\sin^{-1}(n-m)/n] = \sin^{-1}(r-x)/2r$;

therefore $(r-x)/r = 2\sin\frac{1}{3}[\sin^{-1}(n-m)/n]$.

15315. (CONSTANCE I. MARKS, B.A.) [Suggested by a known theorem.]—A series of circles touches a fixed straight line AB at a fixed point C, and a straight line EF parallel to AB cuts each of the series. Show that C lies on one or other of the bisectors of the angles between every possible pair of tangents to the series at the points of section of the circles and EF.

Solutions (I.) *by* V. DANIEL, B.Sc., *and others*; (II.) *by* J. BLAIKIE, M.A.; (III.) *by* H. W. WEBSTER, M.D.; (IV.) *by* B. E. CHATWIN, B.Sc.(Lond.).

(I.) Let one of the series cut EF in P. Draw the tangent PT, meeting AB in T, and the perpendicular CD on EF, and join CP. Then CP

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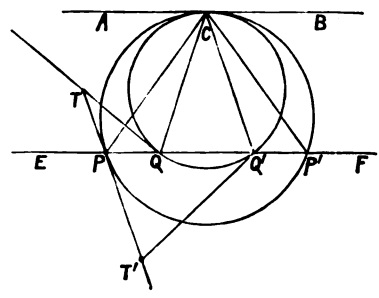
bisects the angle DPT; for TPC is isosceles and $\angle DPC = \angle PCT$, being alternate angles. Hence the perpendicular from C on the tangent at P = CD = constant for all circles of the series, and the angle between every possible pair of such tangents has either its internal or external bisector passing through C.

(II.) Let PCP′, QCQ′ be two members of the series meeting EF in the points P, P′, Q, Q′. Let the tangent at P meet the tangents at Q, Q′ in T, T′. Then



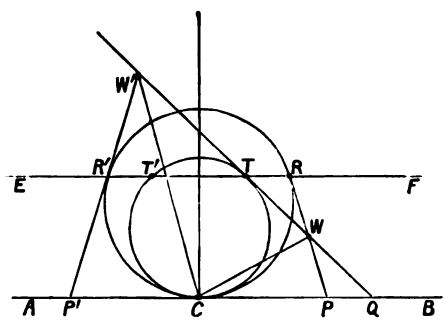
$\angle TPC = \angle CP'P = \angle CPP'$

or PC bisects $\angle TPF$. Similarly CQ bisects the angle TQF and CQ′ bisects the angle T′Q′F. C is therefore an ex-centre of the triangles TPQ, T′PQ′, and similar triangles; therefore CT, CT′ respectively bisect one of the angles made by the tangents which meet at T, T′.

(III.) Take any two circles. Let EF cut them in T, T′ and R, R′ respectively. Draw the tangents at T and R. Let them cut AB in Q and P respectively and each other in W. Then

$$WP/WQ = WR/WT = PR/QT = PC/QC,$$

and, if $WP/WQ = CP/CQ$, the angle W of triangle PWQ is bisected



externally by CW. Similarly, if tangent P′R′ meet QT in W′,

$$W'P'/W'Q = W'R'/W'T = P'R'/QT = P'C/QC,$$

and CW′ bisects the angle W′ of P′W′Q. Similarly with others.

(IV.) Take C as origin, CA as axis of Y, and a line perpendicular to it as axis of X. Let radii of circles be a, b, ... Then the equations of circles are $x^2 - 2ax + y^2 = 0$, $x^2 - 2bx + y^2 = 0$, ... If the equation of EF be $x = c$, the points of intersection of EF and the circles are

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c, $\sqrt{(2ac-c^2)}$; c, $\sqrt{(2bc-c^2)}$; ..., and the equations of the tangents at these points are

$$x(c-a)+y\sqrt{(2ac-c^2)}-ac=0, \quad x(c-b)+y\sqrt{(2bc-c^2)}-bc=0,$$

and the equations of the bisectors are

$$[x(c-a)+y\sqrt{(2ac-c^2)}-ac]/a=\pm[x(c-b)+y\sqrt{(2bc-c^2)}-bc]/b,$$

one of which has no independent term, and hence passes through the origin, *i.e.*, O. [The signs of the roots are of course ambiguous.]

15300. (G. H. Hardy, M.A.)—If $0<\alpha<\frac{1}{2}\pi$, $0<\beta<\frac{1}{2}\pi$, and $0<s<1$,

$$\int_0^{\frac{1}{2}\pi}\cos^{-s}(\theta-\alpha)\cos^{s-2}(\theta-\beta)\,d\theta=\operatorname{cosec}(\beta-\alpha)\left\{\left(\frac{\cos\alpha}{\cos\beta}\right)^{1-s}-\left(\frac{\sin\alpha}{\sin\beta}\right)^{1-s}\right\}.$$

Solution by the Proposer.

Suppose a_1, a_2, c_1, $c_2>0$, and $0<s<1$. Then

$$\frac{\Gamma(s)}{(a_1x+a_1y)^s}=\int_0^\infty e^{-(ax_1+a_2y)t}\,t^{s-1}\,dt,$$

$$\Gamma(s)\int_0^\infty\int_0^\infty\frac{e^{-(c_1x+c_2y)}}{(a_1x+a_2y)^s}\,dx\,dy=\int_0^\infty t^{s-1}\,dt\left(\int_0^\infty\int_0^\infty e^{-(a_1t+c_1)x-(a_2t+c_2)y}\,dx\,dy\right)$$

$$=\int_0^\infty\frac{t^{s-1}\,dt}{(a_1t+c_1)(a_2t+c_2)}$$

$$=\frac{1}{a_1c_2-a_2c_1}\left\{a_1\int_0^\infty\frac{t^{s-1}}{a_1t+c_1}\,dt-a_2\int_0^\infty\frac{t^{s-1}}{a_2t+c_2}\,dt\right\}$$

$$=\frac{\pi}{(a_1c_2-a_2c_1)\sin s\pi}\left\{\left(\frac{a_1}{c_1}\right)^s-\left(\frac{a_2}{c_2}\right)^s\right\}.$$

But the left-hand side is

$$\Gamma(s)\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(a_1\cos\theta+a_2\sin\theta)^s}\int_0^\infty e^{-(c_1\cos\theta+c_2\sin\theta)r}r^{s-1}\,dr$$

$$=\frac{\pi}{\sin s\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(a_1\cos\theta+a_2\sin\theta)^s(c_1\cos\theta+c_2\sin\theta)^{2-s}};$$

so that

$$\int_0^{\frac{1}{2}\pi}(a_1\cos\theta+a_2\sin\theta)^{-s}(c_1\cos\theta+c_2\sin\theta)^{2-s}\,d\theta$$

$$=\frac{1}{a_1c_2-a_2c_1}\left\{\left(\frac{a_1}{c_1}\right)^{1-s}-\left(\frac{a_2}{c_2}\right)^{1-s}\right\}.$$

The formula in the Question follows on putting

$$a_1=\cos\alpha, \quad a_2=\sin\alpha, \quad c_1=\cos\beta, \quad c_2=\sin\beta.$$

15238. (Professor Sanjána, M.A.)—Given of a triangle the base and the vertical angle, construct the triangle so that the ratio may be fixed of the distance of the mid-point of the base from the foot of the perpen-

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dicular drawn from the vertex to its distance from the foot of the bisector of the vertical angle. (A purely geometrical construction is required.)

Solutions (I.) *by* R. F. Davis, M.A., *and* R. Tucker, M.A.; (II.) *by* D. Biddle, T. Savage, B.A., *and others.*

(I.) Let BC be the given base, D its middle point; A the given angle, L and E the points in which its internal bisector intersects the base and the circum-circle respectively; AX the perpendicular on the base. Since $\angle EBC = \angle EAC = \angle EAB$, EB touches the circum-circle of ABL; so that $EB^2 = EL \cdot EA$ and the triangles EBL, EAB are similar.

Thus $EB^2 : EL^2 = EA : EL$. Now $EA : EL = DX : DL$ = known; also $EB^2 : EL^2 = AB^2 : BL^2 = (AB+AC)^2 : BC^2$. Hence $AB+AC$ is known and the solution is easily completed.

(II.) A figure is scarcely needed to show that, if the given base BC be made to form the chord of a circle, which subtends from any point of the circumference (on one side of it) the given angle, and if LDM be the diameter at right angles with such chord, all that is requisite is to find a point on LM such that LX : LD is the given ratio, and through X draw a parallel to the base, cutting the circle in A, the apex, which may be on either side of the vertical diameter.

15288. (Rev. Prebendary W. A. Whitworth, M.A.)—Show that there are *only* five triangles whose sides are integers and whose area is represented by the same integer as the sum of the sides.

Solution by the Proposer.

We must have $4s = (s-a)(s-b)(s-c)$ .... (i.)

or $16(a+b+c) = (b+c-a)(c+a-b)(a+b-c)$. Hence $a+b+c$ must be even, and therefore s integral. From (i.) the factors of s must be contained in $(s-a)(s-b)(s-c)$. Then we may write

$$s = xyz \quad \text{where} \quad 4 = (s-a)/x \,.\, (s-b)/y \,.\, (s-c)/z \ldots\ldots \text{(ii., iii.)}.$$

Of the three fractions in the second member of (iii.) either two each $= 2$ and one $= 1$, or two each $= 1$ and one $= 4$. Thus, either $s-a = 2x$, $s-b = 2y$, $s-c = z$, $s = 2x+2y+z$, $a = 2y+z$, $b = 2x+z$, $c = 2x+2y$; or else $s-a = x$, $s-b = y$, $s-c = 4z$, $s = x+y+4z$, $a = y+4z$, $b = x+4z$, $c = x+y$; and, by (ii.), $xyz = 2x+2y+z$ or $xyz = x+y+4z$.

Note that $x = 2$, $y = 2$ does not give a solution. Excluding this case, unless x or $y = 1$, we have $xy > x+y$; therefore

$$(x+y)z < 2x+2y+z;$$

therefore $(x+y-1)(z-2) < 2$, impossible unless $z = 3$, and this requires x or $y = 1$ or $z = 1$, $x = 7$, $y = 3$; whence the triangle 7, 15, 20; or $(x+y)z < x+y+4z$; therefore $(x+y-4)(z-1) < 4$, impossible unless

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$z = 2$, and this requires x or $y = 1$ or $z = 1$, $x = 6$, $y = 2$; whence the triangle 6, 10, 8.

Therefore the only other solutions occur when $x = 1$ or $y = 1$; and, as x and y are indifferent, it is sufficient to say $x = 1$. Then the solutions are given by $yz = 2y + z + 2$, $(y-1)(z-2) = 4$, whence $y = 5$, $z = 3$ or $y = 3$, $z = 4$ or $y = 2$, $z = 6$, giving the triangles 13, 5, 12; 10, 6, 8; 10, 8, 6; or $yz = y + 4z + 1$, $(y-4)(z-1) = 5$, whence $y = 5$, $z = 6$ or $y = 9$, $z = 2$. These give the triangles 29, 25, 6; 17, 9, 10.

We have thus found seven triangles, of which three are identical, as the only triangles which can satisfy the condition. Hence there are only five different triangles, viz.,

5, 12, 13; 6, 8, 10; 6, 25, 29; 7, 15, 20; 9, 10, 17.

15346. (R. W. D. Christie.)—Establish a connexion between the following pairs of theorems, and, if possible, the general connexion:—

(1)
$$\begin{cases} \cos x + \cos y + \cos z + \cos(x+y+z) \\ \qquad = 4\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(y+z)\cos\tfrac{1}{2}(z+x), \\ x^2 + y^2 + z^2 + (x+y+z)^2 \\ \qquad = (x+y)^2 + (y+z)^2 + (z+x)^2; \end{cases}$$

(2)
$$\begin{cases} \cos(A+B-C) + \cos(A-B+C) + \cos(B+C-A) + \cos(A+B+C) \\ \qquad = 4\cos A\cos B\cos C, \\ (a+b-c)^2 + (a-b+c)^2 + (b+c-a)^2 + (a+b+c)^2 \\ \qquad = 4(a^2+b^2+c^2); \end{cases}$$

(3)
$$\begin{cases} \cos A + \cos B + \cos(A+C) + \cos(B+C) \\ \qquad = 4\cos\tfrac{1}{2}(A+B+C)\cos\tfrac{1}{2}C\cos\tfrac{1}{2}(B-A), \\ a^2 + b^2 + (a+c)^2 + (b+c)^2 \\ \qquad = (a+b+c)^2 + c^2 + (b-a)^2. \end{cases}$$

Solution by Professor Sanjána, M.A.

Expand the cosines in terms of the angles on both sides, and equate the coefficients of like powers of the angular measure; thus a series of identities in terms of the second, fourth, sixth, ... degrees will result.

15364. (R. Tucker, M.A.)—Points D, D′ are taken on the base of the triangle ABC, and equal perpendiculars, DP, D′P′, are let fall upon the sides AB, AC. Similar lines are drawn for the other sides. Find the equation to the conic through the six points. [I propose to call the lines "equal isorthic lines."]

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Solution by Professor SANJÂNA, M.A.

If p is the common length of the perpendiculars, the co-ordinates of D′ are $[0, p, (2\Delta - bp)/c]$ and of D $[0, (2\Delta - cp)/b, p]$. Hence β/γ has the two values $pc/(2\Delta - pb)$, $(2\Delta - pc)/pb$. Putting $\alpha = 0$ in the general conic

$$a'\alpha^2 + b'\beta^2 + c'\gamma^2 - 2f\beta\gamma - 2g\gamma\alpha - 2h\alpha\beta = 0,$$

we get $$b'\beta^2/\gamma^2 - 2f\beta/\gamma + c' = 0;$$

identifying, we have

$$b'/[pb(2\Delta - pb)] = c'/[pc(2\Delta - pc)] = 2f/[p^2bc + (2\Delta - pb)(2\Delta - pc)].$$

So also for the other coefficients; hence the conic is

$$a^2pa(2\Delta - pa) + \beta^2pb(2\Delta - pb) + \gamma^2pc(2\Delta - pc) - \Sigma\beta\gamma[p^2bc + (2\Delta - pb)(2\Delta - pc)] = 0.$$

When lines are drawn through the in-centre parallel to the sides to meet the others, we get for the conic $\Sigma a^2\alpha(b + c) - \Sigma\beta\gamma(a^2 + ab + ac + 2bc) = 0$.

5266. (J. J. WALKER, M.A.)—If the line which is the locus of the equation $lx + my + nz = 0$ meet the sides BC, CA, AB of the triangle of reference in the points A′, B′, C′, and the bisectors of the angles A, B, C in the points D, E, F respectively, prove that

$$m : n = C'A'/C'D : B'A'/B'D, \quad n : l = A'B'/A'E : C'B'/C'E,$$
$$l : m = B'C'/B'F : A'C'/A'F.$$

Solution by Professor NANSON.

The co-ordinates being trilinear, we have

$$m/n = BA'/CA' \cdot CA/AB, \quad B'D/DC' = B'A/AC',$$

$AC'/C'B \cdot BA'/A'C \cdot CB'/B'A = -1$, $AB/BC' \cdot C'A'/A'B \cdot B'C/CA = -1$;

whence $$m : n = C'A'/C'D : B'A'/B'D.$$

15342. (Rev. T. ROACH, M.A.)—Sum the series

$$\frac{1}{1.2.4.5} - \frac{1}{3.4.6.7} + \frac{1}{5.6.8.9} - \ldots.$$

Solutions (I.) *by* Rev. T. WIGGINS, B.A.; (II.) *by the* PROPOSER *and* Professor SANJÂNA, M.A.

(I.) The n-th term is $(-1)^{n-1}\frac{1}{12}\left(\frac{1}{2n-1} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{2n+3}\right)$.

The sum of this series to infinity is easily seen to be

$$\frac{1}{12}\left\{\frac{5}{3} - 2\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right)\right\}, \text{ i.e., } \frac{5}{36} - \frac{1}{6}\log 2.$$

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$$\text{(II.)}\ S = \frac{1}{12}\left\{\left(\frac{1}{1}-\frac{2}{2}+\frac{2}{4}-\frac{1}{5}\right)-\left(\frac{1}{3}-\frac{2}{4}+\frac{2}{6}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{2}{6}+\frac{2}{8}-\frac{1}{9}\right)-\ldots\right\}$$

$$= \frac{1}{12}\left(1-\frac{1}{3}\right)+\frac{1}{6}\left\{-\frac{1}{2}+\frac{1}{4}-\left(-\frac{1}{4}+\frac{1}{6}\right)-\frac{1}{6}+\frac{1}{8}-\left(-\frac{1}{8}+\frac{1}{10}\right)+\ldots\right\}$$

$$= \frac{1}{18}+\frac{1}{6}\left(-\frac{1}{2}+\frac{2}{4}-\frac{2}{6}+\frac{2}{8}-\ldots\right)$$

$$= \frac{1}{18}+\frac{1}{6}\left(\frac{1}{2}-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\ldots\right)$$

$$= \frac{1}{18}+\frac{1}{6}\left(\frac{1}{2}-\log 2\right) = \frac{5}{36}-\frac{1}{6}\log 2.$$

5021. (Professor HUDSON, M.A.)—In a fluid at rest under the action of forces of which $X\delta m$, $Y\delta m$, $Z\delta m$ are the components parallel to the axes acting upon a small mass δm, the locus of points about which the greatest variation of pressure is a maximum or a minimum is determined by

$$\frac{X\frac{d\rho X}{dx}+Y\frac{d\rho X}{dy}+Z\frac{d\rho X}{dz}}{X} = \frac{X\frac{d\rho Y}{dx}+Y\frac{d\rho Y}{dy}+Z\frac{d\rho Y}{dz}}{Y} = \frac{X\frac{d\rho Z}{dx}+Y\frac{d\rho Z}{dy}+Z\frac{d\rho Z}{dz}}{Z},$$

ρ being the density at x, y, z.

Solution by the PROPOSER.

At any given point it is easy to show (indeed it is intuitively evident) that the direction of the greatest variation of pressure is normal to the surface of equal pressure that passes through the point.

This variation of pressure is therefore $l\,\partial p/\partial x + m\,\partial p/\partial y + n\,\partial p/\partial z$, where

$$l^2+m^2+n^2 = 1 \quad \text{and} \quad l/X = m/Y = n/Z.$$

Also $$\partial p/\partial x = \rho X, \quad \partial p/\partial y = \rho Y, \quad \partial p/\partial z = \rho Z,$$

and $\partial(\rho Y)/\partial z = \partial(\rho Z)/\partial y$, $\partial(\rho Z)/\partial x = \partial(\rho X)/\partial z$, $\partial(\rho X)/\partial y = \partial(\rho Y)/\partial x$.

Therefore $l\,\partial p/\partial x + m\,\partial p/\partial y + n\,\partial p/\partial z = \rho(X^2+Y^2+Z^2)^{\frac{1}{2}} = R$, suppose.

Now consider points on a surface of equal pressure; the variation of pressure in passing to an adjacent surface of equal pressure will be a maximum or minimum when

$$\frac{\partial R}{\partial x}dx+\frac{\partial R}{\partial y}dy+\frac{\partial R}{\partial z}\partial z = 0, \quad \text{while} \quad Xdx+Ydy+Zdz = 0;$$

therefore $$\frac{\partial R}{\partial x}\Big/X = \frac{\partial R}{\partial y}\Big/Y = \frac{\partial R}{\partial z}\Big/Z.$$

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Now $$\frac{\partial R}{\partial x} = \left(\rho X \frac{\partial(\rho X)}{\partial x} + \rho Y \frac{\partial(\rho Y)}{\partial x} + \rho Z \frac{\partial(\rho Z)}{\partial x}\right) \Big/ R$$
$$= \rho \left(X \frac{\partial(\rho X)}{\partial x} + Y \frac{\partial(\rho X)}{\partial y} + Z \frac{\partial(\rho X)}{\partial z}\right) \Big/ R;$$

therefore $$\left(X \frac{\partial(\rho X)}{\partial x} + Y \frac{\partial(\rho X)}{\partial y} + Z \frac{\partial(\rho X)}{\partial z}\right) \Big/ X = \text{similar expressions.}$$

If ρ, X, Y, Z are given in terms of x, y, z, these equations determine the lines of greatest and least closeness of the surfaces of equal pressure.

The Calculus of Limits.

By Hugh MacColl, B.A.

The main object of this calculus is to determine the limits of integration in multiple integrals by a purely analytic process wholly independent of all reference to curves or curved surfaces. The ordinary graphic methods are difficult of application when there are three variables, and altogether inapplicable when there are four or more; whereas my analytic method is always applicable. Detached examples of its application have already been given in previous volumes of the *Mathematical Reprint*; my object here is to give a fuller and more connected exposition of the process and of its fundamental principles. For previous examples see *Reprint*, Vol. LXXIII., p. 69; Vol. LXXIV., pp. 49, 59; also, New Series, Vol. I., pp. 81, 87, 88, 97, 111.

(1) Let x be any variable real number or ratio. The symbols x_1, x_2, x_3, ... denote the first, second, third, ... limits of x registered in a Table of Reference (see § 7). There is one limit, however, which occurs so often that it is convenient to represent it *permanently* by the symbol x_0 or y_0 or z_0, according to the particular variable we happen to be considering. This limit is *zero*. Hence $x_0 = y_0 = z_0 = 0$.

(2) The symbol x_m as a *statement* asserts that x_m (the m-th limit of x) is an *inferior* limit, and $x_{m'}$ (with an acute accent over the suffix) asserts that x_m is a *superior* limit of x. The context will always prevent any ambiguity arising from this double signification of the symbol x_m, which sometimes denotes the *limit* x_m, and sometimes the *statement* $x > x_m$. For example, if we have $x_m = a-3$, it is evident that x_m denotes the *limit* $a-3$; whereas, if we have $x_m = (x > a-3)$, x_m is a *statement*. We have also $x_m = (x > x_m)$, and $x_{m'} = (x < x_m)$.

(3) The symbols A^u, A^v respectively assert that A is *positive*, that A is *negative*. From §§ 1, 2 it is evident that $x_{0'}$ (which asserts that zero is a *superior* limit of x) is synonymous with x^v (which asserts that x is negative); and that x_0 (which asserts that zero is an *inferior* limit of x) is synonymous with x^u (which asserts that x is positive).

(4) The symbol $x_{m'.n}$ asserts that x_m is a superior limit and x_n an inferior limit of x. It is therefore short for $(x-x_m)^v(x-x_n)^u$, or its synonym $(x < x_m)(x > x_n)$. The symbol $x_{m'.n'.r.s}$ asserts that x_m and

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x_n are superior limits, and x_r and x_s inferior limits, of x. We thus get the formulæ

$$x_{m'.\,n} = (x-x_m)^{\nu}(x-x_n)^{\mu} = (x<x_m)(x>x_n),$$
$$x_{m'.\,n'.\,r.\,s} = (x-x_m)^{\nu}(x-x_n)^{\nu}(x-x_r)^{\mu}(x-x_s)^{\mu},$$
$$x_{m'.\,0} = (x-x_m)^{\nu}(x-x_0)^{\mu} = (x<x_m)(x>0),$$
$$x_{0'.\,m} = (x-x_0)^{\nu}(x-x_m)^{\mu} = (x-0)^{\nu}(x-x_m)^{\mu}.$$

Similar conventions hold for $y_{m'.\,n}$, $y_{m'.\,0}$, $y_{0'.\,m}$, ...; $z_{m'.\,n}$, $z_{m'.\,n'.\,r.\,0}$; ... .

(5) The following are the three fundamental formulæ of the calculus :—

$$x_{m'.\,n} = x_{m'.\,n}(x_m-x_n)^{\mu} = x_{m'.\,n}(x_m>x_n) \quad \ldots\ldots\ldots\ldots\ldots (1).$$

$$\left.\begin{aligned} x_{m.\,n} &= x_m(x_m-x_n)^{\mu} + x_n(x_n-x_m)^{\mu} \\ x_{m.\,n.\,r} &= x_m\lambda_m + x_n\lambda_n + x_r\lambda_r \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2),$$

in which λ_m, λ_n, λ_r respectively assert that x_m is the nearest of the three *inferior* limits x_m, x_n, x_r; that x_n is the nearest, and so on; so that

$$\lambda_m = (x_m-x_n)^{\mu}(x_m-x_r)^{\mu}, \quad \lambda_n = (x_n-x_m)^{\mu}(x_n-x_r)^{\mu},$$
$$\lambda^r = (x_r-x_m)^{\mu}(x_r-x_n)^{\mu}.$$

$$\left.\begin{aligned} x_{m'.\,n'} &= x_{m'}(x_m-x_n)^{\nu} + x_{n'}(x_n-x_m)^{\nu} \\ x_{m'.\,n'.\,r'} &= x_{m'}\lambda_{m'} + x_{n'}\lambda_{n'} + x_{r'}\lambda_{r'} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (3),$$

in which $\lambda_{n'}$, $\lambda_{n'}$, $\lambda_{r'}$ respectively assert that (of the three *superior* limits x_m, x_n, x_r) x_m is the nearest; that x_n is the nearest, &c.; so that

$$\lambda_{m'} = (x_m-x_n)^{\nu}(x_m-x_r)^{\nu}, \quad \lambda_{n'} = (x_n-x_m)^{\nu}(x_n-x_r)^{\nu},$$
$$\lambda_{r'} = (x_r-x_m)^{\nu}(x_r-x_n)^{\nu}.$$

The symbol η denotes an *impossibility*, the symbol ϵ a *certainty*.

(6) Before I give an example of the application of this "Calculus of Limits," let me explain a symbolic method of abbreviation which I employed for the first time some nineteen years ago in my paper on "The Limits of Multiple Integrals," published in the *Proceedings of the London Mathematical Society*.

The symbols $\phi(x)\,x_{m'.\,n}$ and $x_{m'.\,n}\,\phi(x)$ differ in meaning. The former denotes the integral $\int\phi(x)\,dx$ taken between the superior limit x_m and the inferior limit x_n. The latter means $\phi(x_m)-\phi(x_n)$. For example, suppose $\int\phi(x)\,dx = \psi(x)$. We get

$$\phi(x)\,x_{m'.\,n} = x_{m'.\,n}\,\psi(x) = \psi(x_m)-\psi(x_n).$$

To give a simple example, suppose we have to evaluate the integral $\int_{z_2}^{z_1}dz\int_{y_2}^{y_1}dy\int_{x_0}^{x_1}dx$, the limits being given as in the Table of Reference No. 1. The process fully detailed is as follows :—

Table of Reference No. 1.

$z_1 = y$	$y_1 = x$	$x_1 = a$
$z_2 = c$	$y_2 = b$	$x_0 = 0$

Integral $z_{1'.\,2}\,y_{1'.\,2}\,x_{1'.\,0} = (z_1-z_2)\,y_{1'.\,2}\,x_{1'.\,0}$

$$= (y-c)\,y_{1'.\,2}\,x_{1'.\,0} = y_{1'.\,2}\left(\tfrac{1}{2}y^2-cy\right)x_{1'.\,0}$$
$$= \left\{\left(\tfrac{1}{2}y_1^2-cy_1\right)-\left(\tfrac{1}{2}y_2^2-cy_2\right)\right\}x_{1'.\,0}$$
$$= \left\{\left(\tfrac{1}{2}x^2-cx\right)-\left(\tfrac{1}{2}b^2-cb\right)\right\}x_{1'.\,0}$$
$$= \left(\tfrac{1}{2}x^2-cx-\tfrac{1}{2}b^2+bc\right)x_{1'.\,0}$$
$$= x_{1'.\,0}\left(\tfrac{1}{6}x^3-\tfrac{1}{2}cx^2-\tfrac{1}{2}b^2x+bcx\right)$$
$$= \tfrac{1}{6}a^3-\tfrac{1}{2}ca^2-\tfrac{1}{2}b^2a+bca.$$

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(7) The following problem in probability (Question 14469, Vol. LXXV.) will sufficiently illustrate the working of this "Calculus of Limits":—

Given that a is positive, that n is a positive whole number, and that the variables x and y are each taken at random between a and $-a$, what is the chance that $\{(x+y)^n - a\}$ is negative and $\{(x+y)^{n+1} - a\}$ positive?

Let A denote our data $y_{1'.2}\, x_{1'.2}$ (see Table), let Q denote the proposition $\{(x+y)^n - a\}^v$, and let R denote the proposition $\{(x+y)^{n+1} - a\}^u$ (see § 3).

We have to find the chance $\frac{QR}{A}$, which $= \frac{QRA}{A} = \frac{QRA \int dy \int dx}{A \int dy \int dx}$.

The symbol $\frac{QR}{A}$ denotes the chance that the statements Q and R are true, *assuming* A to be true.

It only remains to find the limits of integration for the numerator from the compound statement QRA, the limits of integration for the denominator being already known, since $A = y_{1'.2}\, x_{1'.2}$. (See the Table of Reference No. 2.)

Table of Reference No. 2.

$y_1 = a$	$x_1 = a$	$a_1 = 1$
$y_2 = -a$	$x_2 = -a$	$a_2 = \frac{1}{2^{n.(n-1)}}$
$y_3 = a^{1/n} - x$	$x_3 = a^{1/n} - a$	
$y_4 = -a^{1/n} - x$	$x_4 = a - a^{1/n} = -x_3$	$a_3 = \frac{1}{2^{(n+1)/n}}$
$y_5 = a^{1.(n+1)} - x$	$x_5 = a + a^{1.(n+1)}$	
$y_6 = -a^{1/(n+1)} - x$	$x_6 = a + a^{1/n}$	
	$x_7 = a^{1/(n+1)} - a$	
	$x_8 = -a - a^{1/(n+1)} = -x_5$	
	$x_9 = a - a^{1/(n+1)} = -x_7$	

Let us take the order of integration y, x. The limits are registered in the table, one after another, as they are found, so that the table *grows* as the process goes on. For convenience of reference, the table should be on a separate slip of paper.

We will first suppose n to be even. Then

$$Q = \{(x+y) - a^{1/n}\}^v \{(x+y) + a^{1/n}\}^u$$
$$= \{y - (a^{1/n} - x)\}^v \{y + (a^{1/n} + x)\}^u = y_{3'.4},$$
$$R = \{(x+y) - a^{1/(n+1)}\}^u = \{y - (a^{1/(n+1)} - x)\}^u = y_5.$$

Hence $QR = y_{3'.4}\, y_5 = y_{3'.4.5}$; and, multiplying by the datum $y_{1'.2}$ (for $A = y_{1'.2}\, x_{1'.2}$), we get (see Note)

$$QR\, y_{1'.2} = y_{3'.1'.2.4.5} = (y_{3'} x_3 + y_{1'} x_{3'})(y_2 x_{4.5} + y_5 x_{5'})$$
$$= (y_{3'} x_3 + y_{1'} x_{3'})(y_2 x_5 + y_5 x_{5'}) = y_{3'.2} x_5 + y_{3'.5} x_{5'.3} + y_{1'.5} x_{3'},$$

for $x_{5.3} = x_5$; $x_{3'.5} = \eta$ (an impossibility); and $x_{3'.5'} = x_{3'}$. These results have been obtained by the formulæ of § 5. (For the detailed working see Note.)

We must now apply formula (1) of § 5 to the expressions in y. We get $y_{3'.2} = y_{3'.2\, x_{6'}}$, $y_{3'.5} = y_{3'.5\, a_1}$, $y_{1'.5} = y_{1'.5\, x_7}$. Substituting these results, we get $QR\, y_{1'.2} = y_{3'.2\, x_{6'}.5} + y_{3'.5\, x_{5'}.3\, a_1} + y_{1'.5\, x_{3'}.7}$.

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We have now found the limits of the variable y, and must apply the three formulæ of § 5 to the expressions in x. Multiplying by the datum $x_{1'.2}$, we get

$$\begin{aligned} \text{QR}\, y_{1'.2}\, x_{1'.2} &= y_{3'.2}\, x_{1'.6'.5.2} + y_{3'.5}\, x_{1'.5'.3.2}\, a_1 + y_{1'.5}\, x_{3'.1'.7.2} \\ &= y_{3'.5}\, x_{1'.3}\, a_1 + y_{1'.5}\, x_{3'.1'.7}, \end{aligned}$$

for $x_{1'.5} = \eta$; $x_{1'.5'} = x_{1'}$; $x_{3.2} = x_3$; $x_{7.2} = x_7$.

We obtain these results immediately by simple inspection of the Table of Reference without having recourse to the formulæ of § 5. Applying the formulæ of § 5 to the expressions in x which remain, we get

$$x_{3'.1'} = x_{3'}\, a_2 + x_{1'}\, a_{2'}\,;\quad x_{1'.3} = x_{1'.3}\, a_2\,;\quad x_{3'.7} = x_{3'.7}\, a_1\,;\quad x_{1'.7} = x_{1'.7}\, a_3.$$

Substituting these values, we get

$$\begin{aligned} \text{QR}\, y_{1'.2}\, x_{1'.2} = \text{QRA} &= y_{3'.5}\, x_{1'.3}\, a_{1.2} + y_{1'.5}\, (x_{3'.7}\, a_{2.1} + x_{1'.7}\, a_{2'.3}) \\ &= y_{3'.5}\, x_{1'.3}\, a_1 + y_{1'.5}\, x_{3'.7}\, a_1 \\ &= (y_{3'.5}\, x_{1'.3} + y_{1'.5}\, x_{3'.7})\, a_1\,; \end{aligned}$$

for $a_{1.2} = a_1 = a_{2.1}$, and $a_{2'.3} = \eta$ (an impossibility).

This is the final step in the process for finding the limits, and the result informs us that, *when n is even*, QRA is only possible when a_1 (that is 1, see Table) is an inferior limit of a. In other words, when n is even and a is not greater than 1, the chance of QR is *zero*. To find the chance when n is even and greater than 1, we have only to evaluate the integrals, employing the abbreviated notation of § 6. Thus

$$\begin{aligned} \text{Integral A} = \text{int}\, y_{1'.2}\, x_{1'.2} &= (y_1 - y_2)\, x_{1'.2} = (2a)\, x_{1'.2} = x_{1'.2}\, (2ax) \\ &= 2ax_1 - 2ax_2 = 4a^2. \end{aligned}$$

$$\begin{aligned} \text{Integral QRA} &= y_{3'.5}\, x_{1'.3} + y_{1'.5}\, x_{3'.7} = (y_3 - y_5)\, x_{1'.3} + (y_1 - y_5)\, x_{3'.7} \\ &= (a^{1/n} - a^{1/(n+1)})\, x_{1'.3} + (a - a^{1/(n+1)} + x)\, x_{3'.7} \\ &= x_{1'.3}\, (a^{1/n} - a^{1/(n+1)})\, x + x_{3'.7}\, (ax - a^{1/(n+1)}\, x + \tfrac{1}{2}x^2) \\ &= (a^{1/n} - a^{1/(n+1)})\, (x_1 - x_3) + (a - a^{1/(n+1)})\, (x_3 - x_7) + \tfrac{1}{2}\, (x_3^2 - x_7^2) \\ &= (a^{1/n} - a^{1/(n+1)})\, (2a - \tfrac{1}{2}a^{1/n} - \tfrac{1}{2}a^{1/(n+1)}) \end{aligned}$$

$$\frac{\text{int QRA}}{\text{int A}} = \frac{\text{int QRA}}{4a^2} = \frac{1}{8a^2}\, (a^{1/n} - a^{1/(n+1)})\, (4a - a^{1/n} - a^{1/(n+1)}) = \frac{\text{QR}}{\text{A}}.$$

We have now to find the chance when n is odd. By the same process as before we get $\text{QRA} = (y_{3'.5}\, x_{1'.3} + y_{1'.5}\, x_{3'.7})\, a_1 + y_{8'.2}\, x_{9'.2}\, a_3$. Here we have *two* inferior limits of a; so that the process is not yet over. To separate the different possible cases, we must multiply the result obtained by the certainty $(a_1 + a_{1'})(a_3 + a_{3'})$, which here reduces to $a_1 + a_{1'.3} + a_{3'}$.

For shortness sake let M_1 denote the bracket coefficient (or co-factor) of a_1 in the result already obtained for QRA, and let M_3 denote $y_{6'.2}\, x_{9'.2}$, the coefficient of a_3. We get

$$\text{QRA} = (\text{M}_1\, a_1 + \text{M}_3\, a_3)(a_1 + a_{1'.3} + a_{3'}) = (\text{M}_1 + \text{M}_3)\, a_1 + \text{M}_3\, a_{1'.3}\,;$$

for $a_{1.3} = a_1$, $a_{3'.1} = \eta$. Hence there are only two possible cases where n is an odd number, the case a_1 (that is to say, $a > a_1$, which means here $a > 1$) and the case $a_{1'.3}$. For the latter $a_{1'.3}$ we get

$$\frac{\text{QR}}{\text{A}} = \frac{\text{int M}_3}{\text{int A}} = \frac{1}{8a^2}\, (2a - a^{1/(n+1)})^2.$$

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For the first case, namely, the case $a > 1$, we get

$$\frac{QR}{A} = \frac{\text{int}\,(M_1 + M_2)}{\text{int}\,A}.$$

When the integrations are worked out we get

$$\frac{QR}{A} = \frac{1}{4a^2}(a^{1/n} - a^{1/(n+1)})(2a - a^{1/n}) + \frac{1}{8a^2}(a^{1/n} - a^{1/(n+1)})^2 + \frac{1}{8a^2}(2a - a^{1/(n+1)})^2.$$

For the case $a < a_3$ the chance is *zero*.

Note.—To show the working of the method, let us take the reduction of the expression $y_{3'.1'.2.4.5}$. We have, by formula (3) of § 5,

$$y_{3'.1'} = y_{3'}(y_3 - y_1)^v + y_{1'}(y_1 - y_3)^v.$$

Now $(y_3 - y_1)^v = (a^{1/n} - x - a)^v = (x - a^{1/n} + a)^u = (x > a^{1/n} - a) = x_3$ and $(y_1 - y_3)^v = x_{3'}$. Substituting, we get $y_{3'.1'} = y_{3'}\,x_3 + y_{1'}\,x_{3'}$.

We must apply formula (2) of § 5 to $y_{2.4.5}$; thus

$$y_{2.4.5} = y_2\lambda_2 + y_4\lambda_4 + y_5\lambda_5,$$

in which

$$\lambda_2 = (y_2 - y_4)^u (y_2 - y_5)^u = (-a + a^{1/n} + x)^u (-a - a^{1/(n+1)} + x)^u$$
$$= (x > a - a^{1/n})(x > a + a^{1/(n+1)}) = x_{4.5} = x_5,$$

$$\lambda_4 = (y_4 - y_2)^u (y_4 - y_5)^u = (-a^{1/n} - x + a)^u (-a^{1/n} - a^{1/(n+1)})^u = \eta \quad \text{(see § 5)},$$

$$\lambda_5 = (y_5 - y_2)^u (y_5 - y_4)^u = (y_5 - y_2)^u \epsilon \text{ (see § 5)} = (y_5 - y_2)^u$$
$$= (a^{1/(n+1)} - x + a)^u = (x - a - a^{1/(n+1)})^v = (x < a + a^{1/(n+1)}) = x_{5'}.$$

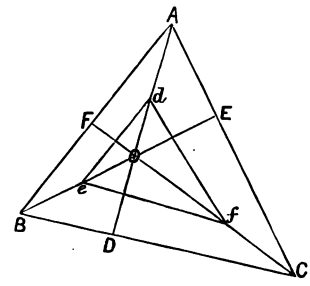
Substituting, we get $y_{2.4.5} = y_2\,x_5 + y_5\,x_{5'}$.

The rest follows from the fact that the limit x_5 is greater than the limit x_3, since $(2a + a^{1/(n+1)} - a^{1/n})$ is always positive.

15333. (R. Tucker, M.A.)—ABC is a triangle; AD, BE, CF are the perpendiculars; and on them are taken (internally) lengths $Ad\,(=l)$, $Be\,(=m)$, $Cf\,(=n)$. Find the area of the triangle *def*. Also find the areas (1) if $l = m = n$; (2) if $l = ka$, $m = kb$, $n = kc$.

Solution by A. W. Poole.

$$AO = 2R\cos A, \ldots;$$
$$\therefore \quad Od = 2R\cos A - l, \ldots;$$
$$\therefore \quad 2\Delta Oef$$
$$= (2R\cos B - m)(2R\cos C - n)\sin A$$
$$= 4R^2 \sin A\cos B\cos C$$
$$- m\,.\,2R\sin A\cos C$$
$$- n\,.\,2R\sin A\cos B + mn\sin A;$$
$$\therefore \quad 2\Delta def$$
$$= 4R^2 \Sigma \sin A\cos B\cos C$$
$$- 2R\Sigma n\,(\sin A\cos B + \cos A\sin B)$$
$$+ \Sigma mn\sin A$$
$$= 4R^2 \sin A\sin B\sin C - 2R\Sigma l\sin A + \Sigma mn\sin A.$$



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(1) If $l = m = n$, then

$$2\Delta def = 4R^2 \sin A \sin B \sin C + (l^2 - 2Rl) \Sigma \sin A$$
$$= 4R^2 \sin A \sin B \sin C + 4 (l^2 - 2Rl) \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C$$
$$= 2\Delta ABC + 4 (l^2 - 2Rl) \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C.$$

(2) If $l = ka$, $m = kb$, $n = kc$, then

$$2\Delta def = 2\Delta ABC - 2kR\Sigma a \sin A + k^2 \Sigma bc \sin A$$
$$= 2\Delta ABC - 2kR\Sigma a \sin A + 6k^2 \Delta ABC.$$

Now $\cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C = \{s/(abc)\}\ \Delta ABC$; therefore in (1) we have

$$2\Delta def = 2\Delta ABC + 4 (l^2 - 2Rl) \{s/(abc)\}\ \Delta ABC;$$

therefore $\Delta def / \Delta ABC = 1 + 2 (l^2 - 2Rl) \{s/(abc)\}$.

Also $\Sigma a \sin A = \Delta ABC \Sigma\ 2a/bc = \{2\Delta ABC/(abc)\}\ \Sigma a^2$;

therefore in (2) we have

$$2\Delta def = 2\Delta ABC - \{4kR\Delta ABC/(abc)\}\ \Sigma a^2 + 6k^2 \Delta ABC;$$

therefore $\Delta def / \Delta ABC = 1 - [2kR (a^2 + b^2 + c^2)]/(abc) + 3k^2$.

14977. (E. Lasker, Ph.D.)—Prove that, if A, B, C, D are four fixed points on a sphere, P a variable one, τ its tangent plane; and if the angle formed by the intersections of PAC, PBC on τ is γ, and the other one formed in the same manner by PAD, PBD is δ, then $\gamma - \delta$ is (in regard to P) a constant magnitude.

Solution by Professor Nanson.

Let A′, B′, C′, D′ be the stereographic projections from P of A, B, C, D; then the trace of PAC on τ is parallel to A′C′; so that γ is equal to A′C′B′, and therefore $\gamma - \delta$ is equal to the difference of the angles A′C′B′, A′D′B′, *i.e.*, to the angle of intersection of the circles A′C′B′, A′D′B′, and therefore to the angle of intersection of the circles ACB, ADB, and this is independent of P.

15372. (Lt.-Col. Allan Cunningham, R.E.)—Prove that 13 is a 2-ic residue of every *prime* Fermat's number $F_n = 2^{2^n} + 1 > 5$. Show how this property could be used as a test of *primality* of F_n.

Solution by the Proposer.

When $n = 0, 1, 2, 3, 4, 5, \ldots$, then $2^n = 1, 2, 4, 8, 16, 256, \ldots$;

therefore $2^{2^n} \equiv 2, 4, 3, 9, 3, 9, \ldots \pmod{13}$;

therefore $F_n \equiv 3, 5, 4, 10, 4, 10, \ldots \pmod{13}$;

therefore $F_n \equiv 4 \pmod{13}$, when x is *even*, and $\equiv 10 \pmod{13}$ when x is *odd* (and $F_n > 5$); therefore F_n is a 2-ic residue of 13 (when $F_n > 5$). Also $(13/F_n) = (F_n/13) = +1$ (when $F_n > 5$); therefore 13 is a 2-ic residue of F_n, when F_n is a *prime* > 5. Hence F_n is expressible in *both* the forms $(t^2 \pm 13u^2)$, when F_n is a prime > 5. Hence, if F_n be > 5, and can be shown to be not expressible in one of the forms $(t^2 \pm 13u^2)$, it must be

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composite. [This property will probably be of no practical use in proving *a priori* the non-primality of any F_n on account of the extreme difficulty of *proving* when F_n is not of form $t^2 \pm 13u^2$.]

15347. (R. Tucker, M.A.)—ABC is a triangle; $\angle BAE = \angle CAD$ and BE, CD, perpendicular to AE, AD respectively, meet in K; then locus of K is a hyperbola about the triangle. Prove that the tangent at A is the "S" line through A, that the fourth point of section is the point in which the join of A to the circum-centre meets the circle, and the three circum-hyperbolas pass through the orthocentre.

Solution by the Proposer *and* Professor Sanjána, M.A.

Equation to AD: $\beta \sin(A-\theta) - \gamma \sin\theta = 0$;
equation to CD: $\alpha \cos\theta + \beta \cos(C+\theta) = 0$;
equation to AE: $\beta \sin\theta - \gamma \sin(A-\theta) = 0$;
equation to BE: $\alpha \cos\theta + \gamma \cos(B+\theta) = 0$.

Therefore K is given by $-\alpha \cos\theta = \beta \cos(C+\theta) = \gamma \cos(B+\theta)$, and the locus is

$$(\alpha + \beta \cos C)\cos\theta = \beta \sin C \sin\theta, \quad (\alpha + \gamma \cos B)\cos\theta = \gamma \sin B \sin\theta,$$

i.e., by $\beta\gamma \sin(B-C) + \gamma\alpha \sin B - \alpha\beta \sin C = 0$ .....(i.),

i.e., a circum-hyperbola; tangent at A is the "S" line through A. Combining (i.) with the circum-circle, we have the fourth point of section given by $-\alpha = \beta \cos C = \gamma \cos B$, *i.e.*, the point where the line AO produced meets the circle (O being the circum-centre).

Now $\beta\gamma \sin(B-C) + \gamma\alpha \sin B - \alpha\beta \sin C = 0$

and $\beta\gamma \sin A + \gamma\alpha \sin(C-A) + \alpha\beta \sin C = 0$

intersect in $\alpha \cos A = \beta \cos B = \gamma \cos C$, *i.e.*, the three hyperbolas pass through the orthocentre.

For the centre we have

$$\frac{\gamma' \sin B - \beta' \sin C}{a} = \frac{\gamma' \sin(B-C) - \alpha' \sin C}{b} = \frac{\beta' \sin(B-C) + \alpha' \sin B}{c};$$

$$\alpha' \propto \begin{vmatrix} a & -\sin C & \sin B \\ b & 0 & \sin(B-C) \\ c & \sin(B-C) & 0 \end{vmatrix} \quad \sin(B-C)[-a \sin(B-C) - c \sin C + b \sin B] = 0,$$

$$\beta' \propto \begin{vmatrix} a & \sin B & 0 \\ b & \sin(B-C) & -\sin C \\ c & 0 & \sin B \end{vmatrix} \quad = \sin B\,[a \sin(B-C) - b \sin B - c \sin C] = -2c \sin B \sin C,$$

$$\gamma' \propto \begin{vmatrix} a & 0 & -\sin C \\ b & -\sin C & 0 \\ c & \sin B & \sin(B-C) \end{vmatrix} \quad = \sin C[-a \sin(B-C) - b \sin B - c \sin C] = -2b \sin B \sin C.$$

Therefore the centre of the hyperbola is the centre of BC.

Analogous results hold for the other two hyperbolas.

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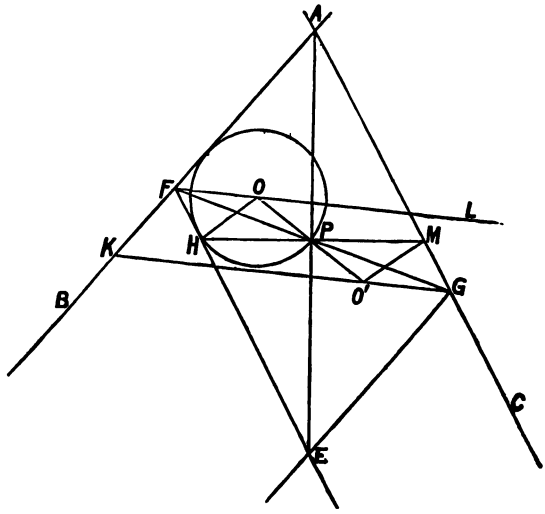
15260. (A. M. Nesbitt, M.A.)—Given a point P within an angle BAC, draw two equal circles through P, one of which shall touch AB and the other AC.

Solution of a more general form of the theorem by J. Blaikie, M.A.

The loci of the centres of circles which pass through P and touch AB or AC are the parabolas which have P as focus and AB or AC as directrix. If, with P as centre and any radius long enough to cut both parabolas, a circle be described, it will in general cut the two parabolas in the four points giving the centres of four equal circles through P, two of which touch AB and the others AC.

Note by George Scott, M.A.

As the theorem proposed in Question 15260 admits of many solutions, I proceed to solve it with the restriction that the two equal circles required shall touch one another at the given point P. The solution is as follows:—Join the vertex A to P. Produce AP to E, making PE = AP. Through E draw parallels to BA and CA, thus completing a parallelogram FAGE. Join the opposite angles F and G. This diagonal will of course pass



through P. By a well known method describe a circle passing through P and touching the lines FA and FE. Call the centre of this circle O. Join O and P to the point of contact on FE, namely, H. Produce HP till it meets AG in M and OP till it meets the bisector of the angle AGE in O'. This last point is the centre of the second required circle, touching AC in M and the first circle in P.

Proof.—O lies on the bisector of the angle AFE, O' on that of AGE. Now these bisectors are parallel, and P may be shown to be equidistant from them; therefore O'P = OP.

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Again, because PM = HP and the vertically opposite angles OPH and O′PM are equal, $\angle O'MP = \angle OHP$; therefore O′M is parallel to OH; therefore O′M is perpendicular to the parallel line AG.

A circle described with O′ as centre and O′P or O′M as radius will touch AC; it will also touch the first circle at P, because the line joining the centres passes through the point P.

15375. (Professor NANSON.)—Prove that

$$(a_1X_2Y_3Z_4)(f_1X_2Y_3Z_4)+(b_1X_2Y_3Z_4)(g_1X_2Y_3Z_4)+(c_1X_2Y_3Z_4)(h_1X_2Y_3Z_4)=0$$

where

$$X_p,\ Y_p,\ Z_p = h_py-g_pz+a_pw,\ f_pz-h_px+b_pw,\ g_px-f_py+c_pw.$$

Solution by THOMAS MUIR, LL.D.

Calling the given expression AF + BG + CH, I note that the performance of the cyclical substitutions $\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$, $\begin{pmatrix} f & g & h \\ g & h & f \end{pmatrix}$, $\begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$ changes A into B, B into C, C into A, F into G, G into H, H into F; and that therefore we need only study the constitution of A and F. Now, by definition,

$$A/w \equiv \begin{vmatrix} 1 & . & . & . & . & . \\ w & . & . & . & -z & y \\ . & w & . & z & . & -x \\ . & . & w & -y & x & . \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2 \\ a_3 & b_3 & c_3 & f_3 & g_3 & h_3 \\ a_4 & b_4 & c_4 & f_4 & g_4 & h_4 \end{vmatrix}$$

and therefore, by the multiplication theorem,

$$= xy\,.\,10 + xz\,.\,9 - y^2\,.\,7 + yz(4-6) + yw\,.\,3 - z^2\,.\,12 - zw\,.\,2,$$

if we denote $|abcf|$, $|abcg|$, $|abch|$ by 1, 2, 3; $|abfg|$, $|bcgh|$, $|cahf|$ by 4, 5, 6; $|abfh|$, $|bcgf|$, $|cahg|$ by 7, 8, 9; $|abgh|$, $|bchf|$, $|cafg|$ by 10, 11, 12; $|afgh|$, $|bfgh|$, $|cfgh|$ by 13, 14, 15, the reason for arranging them in triads being that the performance of the cyclical substitutions changes the first member of any triad into the second member, the second into the third, and the third into the first. Similarly

$$F/w \equiv \begin{vmatrix} . & . & . & 1 & . & . \\ w & . & . & . & -z & y \\ . & w & . & z & . & -x \\ . & . & w & -y & x & . \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2 \\ a_3 & b_3 & c_3 & f_3 & g_3 & h_3 \\ a_4 & b_4 & c_4 & f_4 & g_4 & h_4 \end{vmatrix}$$

$$= -x^2\,.\,13 - xy\,.\,14 - xz\,.\,15 + xw(4+6) - yw\,.\,11 + zw\,.\,8 - w^2\,.\,1.$$

To obviate the necessity of actual multiplication of A by F we next note that

$$F/w = x\{-x\,.\,13 - y\,.\,14 - z\,.\,15\} - w\{-x(4+6)+\ldots\} = x\Omega - wP \text{ (say)},$$

and that therefore $G/w = y\Omega - wQ$ and $H/w = z\Omega - wR$, if Q, R denote

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the expressions obtained by performing the cyclical substitutions on P, Q respectively. It is then seen that $A/w = yR - zQ$; so that

$$B/w = zP - xR \quad \text{and} \quad C/w = xQ - yP.$$

Consequently $(AF + BG + CH)/w^2$ is the same as

$$\begin{vmatrix} x & y \\ P & Q \end{vmatrix} \begin{vmatrix} z & w \\ R & \Omega \end{vmatrix} - \begin{vmatrix} x & z \\ P & R \end{vmatrix} \begin{vmatrix} y & w \\ Q & \Omega \end{vmatrix} + \begin{vmatrix} x & w \\ P & \Omega \end{vmatrix} \begin{vmatrix} y & z \\ Q & R \end{vmatrix},$$

the well known vanishing aggregate connected with the array

$$\begin{matrix} x & y & z & w \\ P & Q & R & \Omega. \end{matrix}$$

N.B.—The writing of A, B, ... as quasi-products suggests a generalization which is formulated elsewhere (see Question 15426). The case of next lower order is

$$\begin{vmatrix} 1 & . & . & . \\ y & . & . & x \\ . & y & -x & . \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \times \begin{vmatrix} . & . & 1 & . \\ y & . & . & x \\ . & y & -x & . \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

$$+ \begin{vmatrix} . & 1 & . & . \\ y & . & . & x \\ . & y & -x & . \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} \times \begin{vmatrix} . & . & . & 1 \\ y & . & . & x \\ . & y & -x & . \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0.$$

15358. (C. M. Ross.)—Find the coefficient of $x^m y^n$ in the expansion of $[(1+x)^p (1+y)^p]/(1+x+y)^p$.

Solution by J. M. Allen, F.I.A.

$$\left\{\frac{(1+x)(1+y)}{1+x+y}\right\} = \left\{1 + \frac{xy}{1+x+y}\right\}^p$$

$$= 1 + p\frac{xy}{1+x+y} + \frac{p(p-1)}{2!}\left(\frac{xy}{1+x+y}\right)^2 + \ldots,$$

the general term being $\{[p(p-1)\ldots(p-r+1)]/r!\}\, x^r y^r\, (1+x+y)^{-r}$.

We have now to find the coefficient of $x^{m-r}y^{n-r}$ in the expansion of $(1+x+y)^{-r}$, which consists of a series of ascending powers of $(x+y)$. In this expansion $x^{m-r}y^{n-r}$ can evidently only occur in the $(m+n-2r)$-th power of $(x+y)$, and its coefficient there is $(m+n-2r)!/(m-r)!\,(n-r)!$. But the coefficient of $(x+y)^{m+n-2r}$ in the expansion of $(1+x+y)^{-r}$ is

$$(-1)^{m+n-2r}[r(r+1)\ldots(m+n-r-1)]/(m+n-2r)!.$$

Now, because r cannot be fractional, $(-1)^{m+n-2r}$ must be $= (-1)^{m+n}$, and we get for the coefficient of $x^{m-r}y^{n-r}$ in the expansion of $(1+x+y)^{-r}$: $(-1)^{m+n}[r(r+1)\ldots(m+n-r-1)]/(m-r)!\,(n-r)!$.
Thus we get for the total coefficient of $x^m y^n$ in the original expansion:

$$(-1)^{m+n} \Sigma \left[\frac{\{p(p-1)\ldots(p-r+1)\}\{r(r+1)\ldots(m+n-r-1)\}}{r!\,(m-r)!\,(n-r)!}\right].$$

The upper limit of the summation will, of course, be p if p be a positive integer not greater than either m or n.

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15286. (R. W. D. Christie.)—Extend Euler's theorem of two squares to the cubes, *e.g.*, $(1^3+19^3)(17^3+37^3) = 683^3+397^3$.

Solution by Lt.-Col. Allan Cunningham, R.E.

Let $n = (x^3+y^3)+(x+y)$, $N = (X^3+Y^3)+(X+Y)$; n, N both *odd*. Then $n = (x^2-xy+y^2)$ is expressible in the form $n = (t^2+3u^2)$ in *one* of the *three* following ways, according as x, y are *odd* or *even* (even if n be prime):—

$$n = (\tfrac{1}{2}x \sim y)^2 + 3(\tfrac{1}{2}x)^2, \text{ if } x \text{ is } even \quad \ldots\ldots (1)$$
$$= (x \sim \tfrac{1}{2}y)^2 + 3(\tfrac{1}{2}y)^2, \text{ if } y \text{ is } even \quad \ldots\ldots (2)$$
$$= \{\tfrac{1}{2}(x+y)\}^2 + 3\{\tfrac{1}{2}(x-y)\}^2, \text{ if } x, y \text{ are both } odd \quad \ldots (3).$$

Similarly $N = (X^2-XY+Y^2)$ is expressible in the form $N = (T^2+3U^2)$ in *one* of three ways, similar to the above, according as X, Y are *odd* or *even* (even if N be *prime*). And the product $nN = (t^2+3u^2)(T^2+3U^2)$ is expressible in *two* ways in the form $nN = A^2+3B^2$ given by

$$A = (tT \mp 3uU), \quad B = tU \pm uT \quad \ldots\ldots\ldots\ldots (4).$$

Again (A^2+3B^2) may be expressed in the form $(R^3+S^3)\div(R+S)$ in the *three* following ways, derivable from (1), (2), (3) above:—

$$A^2+3B^2 = \{(2B)^2+(B-A)^2\}+\{2B+(B-A)\}, \text{ if } B>A \quad \ldots\ldots (5)$$
$$= \{(A+B)^2+(2B)^2\}+\{(A+B)+B\} \quad \ldots\ldots (6)$$
$$= \{(A+B)^2+(A-B)^2\}+\{(A+B)+(A-B)\}, \text{ if } A>B \quad (7).$$

Hence $$nN = \frac{x^3+y^3}{x+y}\,\frac{X^3+Y^3}{X+Y} = \frac{R^3+S^3}{R+S}.$$

This gives the required partition

$$(x^3+y^3)(X^3+Y^3) = R^3+S^3 \quad \ldots\ldots (8),$$

provided

$$(x+y)(X+Y) = R+S \quad \ldots\ldots (9).$$

This condition (9) determines the particular form (5), (6), (7), as it reduces to *either* of the following:—

$$(x+y)(X+Y) = 3B-A, \text{ if } B>A \quad \ldots\ldots (5a)$$
$$= A+2B \quad \ldots\ldots (6a)$$
$$= 2A, \text{ if } A>B \quad \ldots\ldots (7a).$$

These conditions (5*a*), (6*a*), (7*a*) can of course be expressed in terms of the original x, y, X, Y; but the algebraical work is rather tedious, and would take a good deal of space. The numerical work with given numbers is far simpler.

E.g., given $(x^3+y^3)(X^3+Y^3) = (1^3+19^3)(17^3+37^3)$. By result (3),
$n = 10^2+3\cdot 9^2 = t^2+3u^2$, $N = 27^2+3\cdot 10^2 = T^2+3U^2$.

By result (4), $A = 10\cdot 27 \mp 3\cdot 9\cdot 10$, $B = 10\cdot 10 \pm 243$; therefore

$$nN = 540^2+3\cdot 143^2 = 0^2+3\cdot 343^2 = A^2+3B^2.$$

Here condition (7*a*) is satisfied, since

$$(x+y)(X+Y) = 20\cdot 54 = 1080 = 2A,$$

thus showing that form (7) is to be used, giving, by (8),

$$(x^3+y^3)(X^3+Y^3) = (A+B)^3+(A-B)^3 = 683^3+397^3,$$

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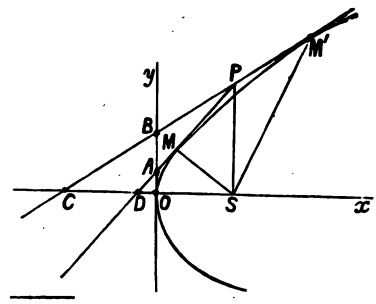
15120. (Professor Cochez.) —On considère la parabole

$$y^2 = 2px.$$

Lieu des points P tels que, menant les tangentes PM, PM′ rencontrant les axes en A, D, B, C, le quadrilatère ABCD soit circonscriptible.



Solutions (I.) *by* Professor Sanjána, M.A., *and* W. Scrimgeour, M.A., B.Sc.; (II.) *by* J. H. Taylor, M.A., *and* A. W. Poole; (III.) *by* V. Daniel, B.Sc.

(I.) Since ABCD is cyclic,

$$\angle ADO = \angle ABC = 90° - \angle BCO = 90° - \angle BM'S = 90° - \angle MPS;$$

therefore SP is at right angles to Ox. Thus the locus of P is the latus rectum.

(II.) The equations of the tangents at M and M′,

$$y = mx + a/m, \quad y = m_1x + a/m_1 \ldots\ldots\ldots\ldots\ldots (1, 2),$$

mark the vertex O.

Then $AO = a/m$, $BO = a/m_1$, $CO = -a/m_1^2$, $DO = -a/m^2$. By hypothesis $OB.OA = OC.OD$, whence $mm_1 = 1$, from (1) and (2). Abscissa of $P = a/mm_1 = a$. Therefore the locus is the part outside the curve of the double ordinate through the focus.

The converse of this Question is a very simple geometrical problem.

(III.) If P be the point (x', y'), the tangents are $y - m_1x - p/2m_1 = 0$ and $y - m_2x - p/2m_2$, where the m's satisfy the quadratic $2m^2x' - 2my' + p = 0$. Any conic through the intersections of the tangents from P with the axes is given by $\lambda xy + (y - m_1x - p/2m_1)(y - m_2x - p/2m_2) = 0$.

For this to be a circle we must have $m_1m_2 = 1$, $m_1 + m_2 = \lambda$, *i.e.* $2x' = p$, $y'/x' = \lambda$. Hence the locus is the latus rectum.

15222. (I. Arnold.)—There are two inclined planes each 100 feet long, viz., AC and BC, and each inclined at an angle of 15° to the horizon. A small smooth weight slides freely down AC and then ascends BC; after descending BC it reascends AC; and so on. It is required to find the number of times it will ascend the plane AC before losing half the velocity acquired in the first descent.

Solution by Constance I. Marks, B.A.

Let v be the velocity acquired in the first descent; then $v \cos 30°$ is the velocity up the plane of ascent at the commencement of the first ascent,

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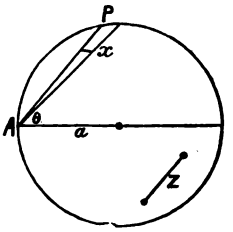
and the velocity up the plane of ascent at the commencement of any ascent up either plane is evidently that at the beginning of the previous ascent up the other plane multiplied by cos 30°.

Now let $\cos^n 30° = x$; therefore $n \log \frac{1}{2}\sqrt{3} = \log x$, that is, $\frac{1}{2}n \log 3 - n \log 2 = \log x$.

Also $\log \frac{1}{2} = -\log 2 = -\cdot 3010300$; therefore the velocity will be reduced by one-half or more than one-half, respectively, as soon as $\frac{1}{2}n \log 3 - n \log 2$ becomes algebraically equal to or less than $-\cdot 3010300$. Putting $n = 1, 2, 3, \ldots$ in succession in this expression, the velocity at the beginning of the fourth ascent—that is, the second up AC—is at once seen to be greater than $\frac{1}{2}v$, but at the beginning of the fifth ascent or the third up BC it is less than $\frac{1}{2}v$; therefore the weight will ascend AC twice before it loses half the velocity acquired in the first descent.

15377. (R. Chartres.)—Find the mean value of Z^n when Z is the distance between two random points (1) in a circle, (2) in a sphere; and in the latter case, if $M(Z^n) = p_n a^n$, find m and n (i.) when $p_n = p_m$, (ii.) when $p_n = 2p_m$, a being the radius of the sphere.

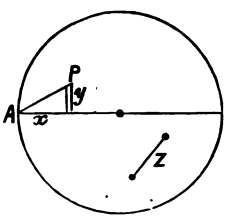
Solution by the Proposer.



(1) $$M(x) = \frac{2}{n+2}\int x^n \tfrac{1}{2}x^2 d\theta / \tfrac{1}{2}\pi a^2$$
$$= \frac{2^{n+3}a^n}{(n+2)\pi}\int_0^{\frac{1}{2}\pi} \cos^{n+2}\theta \, d\theta.$$

Multiplying by $4/(n+4)$,

$$M(Z^n) = \frac{2^{n+5}a^n}{(n+2)(n+4)\pi}\int_0^{\frac{1}{2}\pi} \cos^{n+2}\theta \, d\theta.$$



(2) $$M(AP^n)$$
$$= \int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}} (x^2+y^2)^{\frac{1}{2}n} 2\pi y \, dx \, dy / \tfrac{4}{3}\pi a^3$$
$$= \frac{3.2^{n+2}a^n}{(n+3)(n+4)}.$$

Multiplying by $6/(n+6)$,

$$M(Z^n) = \frac{9.2^{n+3}a^n}{(n+3)(n+4)(n+6)}.$$

If $n = 2$, $M_2 = \frac{6}{5}a^2$; if $n = -1$, $M_{-1} = \frac{9}{8}a^{-1}$; if $n = 1$, $M_1 = \frac{32}{35}a$; if $n = 4$, $M_4 = \frac{72}{55}a^4$.

15354. (Robt. W. D. Christie.)—Let P be any prime of form $4m+1$, and r any primitive root; then $r^m(r^m+1) = r^m - 1 \pmod{P}$. Show the utility of the sequence in arriving at the first prime root.

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Solution by the PROPOSER.

For all primes of form $4m+1$ we have, by the theorem of primitive roots,

$$r^{4m} = 1 \pmod{p},$$

i.e., $r^{2m}+1 = 0 \pmod{p}$ necessarily, *i.e.*, $r^m(r^m+1)-(r^m-1) = 0 \pmod{p}$; hence the theorem.

Similarly $$r^{3m}(r^{3m}+1)-(r^{3m}-1) = 0 \pmod{p}$$

or $$r^{3m}(r^{3m}+1) = r^{3m}-1 \pmod{p},$$

the cognate sequence.

Note.— $(r^m + r^{3m}) = 0 \pmod{p}$. To utilize the sequence first get r^m for any prime of given form, *e.g.*, 313. Here $p = 13^2+12^2$; therefore $\pm r^m = \mp r^{3m} = 25$. Thus the two sequences are

$$25.26 = 24 \pmod{313}, \quad 288.289 = 287 \pmod{313}.$$

To utilize r^m still further, let r^m or $r^{3m} = x$. Then, since $x^{2n} = \mp 1 \pmod{p}$ for all values of n, we may factorize any such expression as x^4-x^2-2 or a^2+b^2+2, &c., *ad lib.*; *e.g.*, let $p = 44537$, then $11187^4-11187^2-2 = 0 \pmod{44537}$. Now, by resolving the prime into its integral squares, we easily get $\pm r^m$ by the G.C.M., or, if x be given, the *prime* is recoverable from x^2+1.

15277. (Professor COCHEZ.)—Étudier les courbes

(1) $y^2 = x^4/(1-x)$, (2) $y = 3x+5+(x-2)/(x^2-8x+15)$.

Solution by A. M. NESBITT, M.A.

(1) $y^2 = x^4/(1-x)$, asymptote $x = 1$. Approximately at origin $y = \pm x^2$, at infinity $y^2 = -x^3$, $dy/dx = y(4-3x)/2x(1-x)$. Any number of points may readily be found on the curve, such as $x = -3, -8$, $y = \pm\frac{9}{2}, \pm\frac{64}{3}, \ldots$, $dy/dx \mp \frac{39}{16}, \mp\frac{49}{27}$. Hence the curve may be approximately drawn.

(2) $y = 3x+5+(x-2)/\{(x-3)(x-5)\}$, asymptotes $x = 3$, $x = 5$, $y = 3x+5$, $dy/dx = 3-(x^2-4x+1)/\{(x-3)^2(x-5)^2\}$. Hence, again, any number of points may be found on the curve; and it will be seen that the tangent is parallel to the x axis between $x = 5\frac{1}{2}$ and $x = 6$, the corresponding values for y being $24\frac{1}{6}$ and $24\frac{1}{3}$, and those for dy/dx $-\frac{39}{1152}$ and $+\frac{14}{9}$; and the same thing happens between $x = 4{\cdot}3$ and $x = 4{\cdot}\dot{3}$, the values of y being 15·36 and 15·375, while dy/dx has values ·2 (approximately) and $-\frac{3}{32}$. The curve may now be approximately drawn.

15405. (J. MACLEOD, H.M.I.S., I.S.O.)—From the point O a tangent OQ is drawn to the point Q of the hyperbola whose focus is S. A circle is described about the triangle OQS, and in OQ a point P is taken such

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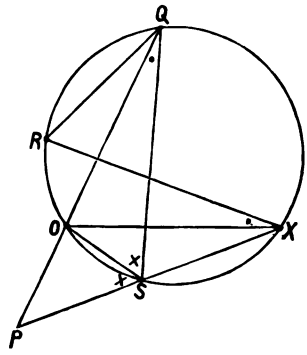
that QS : QO = SP : PO. SP being produced to meet the circle in X, at X in OX the angle OXR (R being on the circle) is formed equal to the angle OQS. Prove that RX = the focal distance of Q and that the focus S′ lies on QR.

Solutions (I.) *by* R. F. Davis, M.A., *the* Proposer, *and many others;*
(II.) *by* J. Blaikie, M.A.



(I.) Since SO bisects externally the angle QSX,

arc ORQ = arc OSX.

Similarly, since

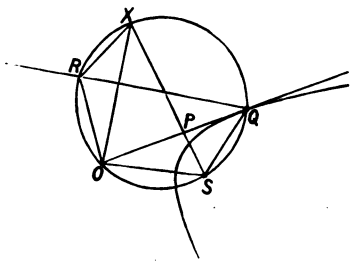
$\angle$ OXR = $\angle$ OQS,

arc OR = arc OS.

Hence arc RQ = arc SX,

and chord RX = chord SQ.

Also, since the angle SQR is bisected by QO, QR passes through S′.



(II.) We have

QS/QO = SP/PO;

therefore QS/SP = QO/PO;

but QS/SP = XO/PO,

by similar triangles QSP, XOP;

therefore QO = XO;

therefore $\angle$ QSO = $\angle$ XRO,

and $\triangle$QSO $\equiv$ $\triangle$XRO,

by *Euc.* I. 26;

therefore RX = SQ.

Also $\angle$ RXO = $\angle$ OQS

and $\angle$ RXO = $\angle$ RQO;

therefore RQO = OQS, and QO is a tangent; therefore QR goes through the other focus.

[Mr. F. W. Reeves, B.A., points out that the proposition is true for the ellipse as well as for the hyperbola.]

15285. (Lt.-Col. Allan Cunningham, R.E.)—If y, Y are *proper* roots of

$$(y^n-1)+(y-1)\equiv 0 \pmod p,$$

and y', Y′ are *proper* roots of

$$(y'^n+1)+(y'+1)\equiv 0 \pmod p,$$

and, if y, y' are $<\frac{1}{2}p$, and Y, Y′ are $>\frac{1}{2}p$, but $<p$, show that (when p, q, r are unequal *odd* primes) then

i. $\Sigma y-\Sigma y' = \Sigma Y-\Sigma Y' \equiv -1$, if $n=q$;

and ii. $\equiv 0$, if $n=q^2$;

and iii. $\equiv +1$, if $n=qr$;

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the congruences being all for modulus p, and the summations including all incongruous roots within the limits stated.

Solution by the PROPOSER.

This Question is a corollary of Question 15226, and follows at once from the solution thereof.

For, since any root $r\,(<\frac{1}{2}p)$ or $\mathrm{R}\,(>\frac{1}{2}p)$ of either congruence gives a root $(p-r)\,(>\frac{1}{2}p)$ or $(p-\mathrm{R})\,(<\frac{1}{2}p)$ of the *conjugate* congruence, therefore the set of roots y with the set $(p-\mathrm{Y}')$ are a *complete* set; also the set of roots Y with the set $(p-\mathrm{Y}')$ are a *complete* set of the incongruous roots $(<p)$ of the *first* congruence. Hence, by the solution of Question 15226,

i. $\Sigma y+\Sigma(p-y')\equiv -1$, $\Sigma(p-\mathrm{Y}')+\Sigma\mathrm{Y}\equiv -1$ when $n=q$;

ii. Both above $\equiv 0$, if $n=q^2$;

iii. Both above $\equiv +1$, if $n=qr$.

As $\Sigma(p)$ vanishes to mod p, the above results reduce to those required.

15350. (A. M. NESBITT, M.A.)—On the tangent at P to an ellipse a point T is taken, such that TP = TC; TR and TP are equally inclined to the axis, and CR is drawn perpendicular to TR. If CQ be conjugate to CP, prove that the ratio CR : CQ is constant.

Solution by the PROPOSER.

Let θ be the eccentric angle of P.

$\mathrm{TC}=\mathrm{TP}=\mathrm{M}$,

$\angle\mathrm{TCP}=\angle\mathrm{TPC}=\phi$,

$\angle\mathrm{TNM}=\angle\mathrm{TMN}=\psi$,

$\mathrm{CP}=a'$, $\mathrm{CQ}=b'$.

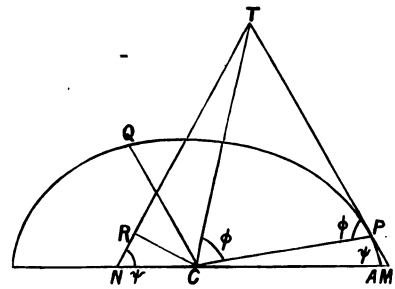
Then, since

$\tan\psi=b/a\cot\theta$,

and co-ordinates of Q are $-a\sin\theta$, $b\cos\theta$, we have

$\cos\psi/a\sin\theta=\sin\psi/b\cos\theta$

$=1/b'$;



also $\cos\phi=\cos(\psi+\mathrm{PCA})$; therefore $\cos\phi=[(a^2-b^2)\cos\theta\sin\theta]/a'b'$

whence $2r=a'\sec\phi=a'^2b'/[(a^2-b^2)\sin\theta\cos\theta]$. Again

$$\angle\mathrm{CTR}=\angle\mathrm{TCA}-\psi=(\phi+\angle\mathrm{PCA})-(\phi-\angle\mathrm{PCA})=2\angle\mathrm{PCA};$$

therefore $\mathrm{CR}=2r\sin\mathrm{PCA}\cos\mathrm{PCA}=2r\dfrac{ab\sin\theta\cos\theta}{a'^2}=\dfrac{abb'}{a^2-b^2}$;

therefore $\mathrm{CR}:\mathrm{CQ}=ab/(a^2-b^2)=\sqrt{(1-e^2)}/e^2$.

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15343. (A. M. Nesbitt, M.A.)—If

$$x+y+z=0 \quad \text{and} \quad ax^2+by^2+cz^2=0,$$

prove that, if λ^2 be a certain rational function of a, b, c,

$$[(a+\lambda)^n-(a-\lambda)^n]x^n+[(b+\lambda)_n-(b-\lambda)^n]y^n+[(c+\lambda)^n-(c-\lambda)^n]z^n=0.$$

Solution by the Proposer.

Put $x=q-r$, $y=r-p$, $z=p-q$; then, by the second equation, we may take $ax=\lambda p+\mu$, $by=\lambda q+\mu$, $cz=\lambda r+\mu$; whence at once

$$(a+\lambda)x=\lambda(p+q-r)+\mu=by-\lambda y=(b-\lambda)y.$$

So $$(b+\lambda)y=(c-\lambda)z, \quad (c+\lambda)z=(a-\lambda)x.$$

Hence, by multiplication, $(a+\lambda)(b+\lambda)(c+\lambda)=(a-\lambda)(b-\lambda)(c-\lambda)$ or $\lambda^2+bc+ca+ab=0$, and plainly $\Sigma\{(a+\lambda)x\}_n=\Sigma\{(a-\lambda)x\}^n$.

15325. (Salutation.)—There is a two-figured number (A) which doubled less 1 becomes reversed; if (A) doubled less 10 be taken from thrice a three-figured number (B), the latter is reversed; and if to (B) be added one-tenth of its value less 1 (remainders being rejected), and the whole be deducted from a four-figured number (C) quadrupled, then (C) is reversed. Find (A), (B), (C).

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E., *and another;*
(II.) *by* Rev. T. Wiggins *and others.*

(I.) Let the digits of each number in turn be denoted by b, a for A; c, b, a for B; d, c, b, a for C; a being the units digits in each case; then $a, b, c, d \not> 9$, $a \not< 1$. Then $A=10b+a$ and $2(10b+a)-1=10a+b$; Therefore $19b=8a+1$; therefore $b=3$, $a=7$; $A=37$.

Next, $B=100c+10b+a$ and $3(100c+10b+a)-64=100a+10b+c$; therefore

$$299c+20b=97a+64.$$

By taking $a=1, 2, 3, \ldots, 9$ in turn, it is soon found that the *only* solution is $c=3$, $b=2$, $a=9$; $B=329$.

Lastly, $$C=1000d+100c+10b+a$$

and $$4(1000d+100c+10b+a)-360=1000a+100b+10c+d;$$

therefore $$3999d-996a=60b+360-390c;$$

therefore $$1333d-332a=10(2b+12-13c).$$

And, since $1333d-332a\equiv 0 \pmod{10}$, therefore $3d=2a$; hence, if $d=2$; 4; 6; 8, $a=3$, 8; 1, 6; 4, 9; 2, 7 and

$$13c=2b+12+\tfrac{1}{10}(332a-1333d)=2b+12+\tfrac{1}{10}\{332(a-4d)-5d\}.$$

Here, since $a, b, c, d \not> 9$, this requires $a-4d=0$, so that (by previous result) $a=8$, $d=2$, $13c=2b+12-1$, of which the *only* solution is $b=1$, $c=1$, giving $c=2118$.

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(II.) (1) To find A. Let it be $(10x+y)$; therefore

$$2(10x+y)-1 = (10y+x) \text{ or } (19x-1) = 8y;$$

whence $x<4$; and on testing $x=3$; therefore A is 37.

(2) To find B. Let it be $(100x+10y+z)$; therefore

$$3(100x+10y+z)-64 = (100z+10y+x);$$

therefore $$299x+20y = 97z+64$$

or $$97(3x-z) = 64-20y-8x = 4(16-5y-2x);$$

whence obviously $3x-z=0$, $y<3$; and on testing the number is found to be 329.

(3) To find C. Let it be $(1000x+100y+10z+a)$; therefore

$$4(1000x+100y+10z+a)-360 = 1000a+100z+10y+x.$$

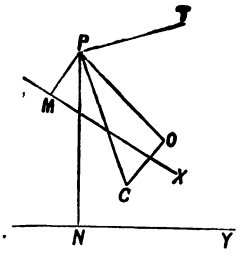
Obviously $x<3$; $4a-x$ is multiple of 10; therefore $x=2$, $a=8$ or 3. Clearly the greater value of a must be taken; therefore on substituting and reducing $13y = 2z+11$; whence $y=1$, $z=1$; therefore $c=2118$.

Hence A = 37, B = 329, C = 2118.

15401. (Professor L. Catalá.)—Décrire une circonférence qui soit tangente à une droite ou à une circonférence en un point donné et qui passe par le point d'intersection inaccessible de deux droites données.

Solution by Rev. T. Wiggins.

Let P be the given point of the given line PT or given circumference. Let PM, PN be the perpendiculars on the given lines. Let O be the centre of the circle PMN. Because PMX, PMY are right angles, PO produced passes through the inaccessible point. Draw OC perpendicular to OP, and PC perpendicular to PT. Then C is the required centre; for the circle with C as centre and CP as radius touches the given tangent at the given point and cuts PO again, where it is cut by the circle PMN, *i.e.*, passes through the inaccessible point.



9656. (J. P. Johnston, B.A.)—The four lines joining the points where the Hessian of a cubic surface touches three coplanar lines in the surface intersect the Hessian again in four collinear points.

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Solution by H. M. TAYLOR, M.A., F.R.S.

If the cubic surface passes through the three edges BC, CA, AB of the tetrahedron of reference ABCD, the equation of the cubic in tetrahedral co-ordinates may be written

$$u(ax^2+by^2+cz^2+du^2+2lxu+2myu+2nzu+2pyz+2qzx+2rxy)-2xyz=0.$$

The intersection of the Hessian of the cubic by the plane $u=0$ is given by the equation

$$\begin{vmatrix} . & -z & -y & \mathrm{L} \\ -z & . & -x & \mathrm{M} \\ -y & -x & . & \mathrm{N} \\ \mathrm{L} & \mathrm{M} & \mathrm{N} & \mathrm{P} \end{vmatrix} = 0$$

where $\mathrm{L}=ax+ry+qz$, $\mathrm{M}=rx+by+pz$, $\mathrm{N}=qx+py+cz$,

and $$\mathrm{P}=lx+my+nz.$$

The Hessian touches the line BC where it meets

$$z(py+cz)-y(by+pz)=by^2-cz^2=0,$$

and similarly for the lines CA, AB. Hence the six points of contact of the Hessian with the sides of the triangle ABC lie on the four lines represented by the equation $a^{\frac{1}{2}}x+b^{\frac{1}{2}}y+c^{\frac{1}{2}}z=0$, if proper changes be made in the signs of the radicals.

The above determinant may be transformed into the form

$$\begin{vmatrix} . & -z & -y & ax \\ -z & . & -x & by \\ -y & -x & . & cz \\ \mathrm{L} & \mathrm{M} & \mathrm{N} & \mathrm{Q} \end{vmatrix}$$

where $\mathrm{Q}=\mathrm{P}+p\mathrm{L}+q\mathrm{M}+r\mathrm{N}$, and into the form

$$\begin{vmatrix} . & -z & -y & a^{\frac{1}{2}}\mathrm{U} \\ -z & . & -x & b^{\frac{1}{2}} \\ -y & -x & . & c^{\frac{1}{2}}\mathrm{U} \\ \mathrm{L} & \mathrm{M} & \mathrm{N} & \mathrm{R} \end{vmatrix}$$

where $\mathrm{U}=a^{\frac{1}{2}}x+b^{\frac{1}{2}}y+c^{\frac{1}{2}}z$ and

$$\mathrm{R}=\mathrm{Q}-(bc)^{\frac{1}{2}}\mathrm{L}-(ca)^{\frac{1}{2}}\mathrm{M}-(ab)^{\frac{1}{2}}\mathrm{N}$$
$$=(l+ap+2qr)x+(m+bq+2pr)y+(n+cr+2pq)z-(a^{\frac{1}{2}}p+b^{\frac{1}{2}}q+c^{\frac{1}{2}}r)\mathrm{U}.$$

Hence any of the four lines represented by the equation $a^{\frac{1}{2}}x+b^{\frac{1}{2}}y+c^{\frac{1}{2}}z=0$ meets the Hessian in the line $u=0$,

$$(l+ap+2qr)x+(m+bq+2pr)y+(n+cr+2pq)z=0.$$

15332. (A. M. NESBITT, M.A.)—Any tangent to a hyperbola meets the asymptotes in P and P′. Prove that the locus of the centre of the circle circumscribing the triangle CPP′ is a hyperbola.

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Solutions (I.) *by* R. TUCKER, M.A., *and the* PROPOSER;
(II.) *by* Rev. T. WIGGINS, B.A., *and* Professor SANJÂNA, M.A.

(I.) With the usual notation, let (h, k) be the co-ordinates of the point of contact; then the equations to the two perpendiculars through (h, o) and (o, k) are $x + y\cos\omega = k$, $y + x\cos\omega = h$; therefore the locus is

$$xy(1+\cos^2\omega) - (x^2+y^2)\cos\omega = hk = 4c^2.$$

(II.) Let the equation of the given hyperbola be $x^2/a^2 - y^2/b^2 = 1$. Then the tangent at (x', y') is $xx'/a^2 - yy'/b^2 = 1$. Let this tangent meet the asymptotes $[y = \pm(b/a)x]$ at (x_1, y_1), (x_{11}, y_{11}). Therefore

$$x_1 = \frac{a}{x'/a - y'/b},\ y_1 = \frac{b}{x'/a - y'/b};\quad x_{11} = \frac{a}{x'/a + y'/b},\ y_{11} = \frac{b}{-(x'/a + y'/b)} \quad \text{(i.)}.$$

The line bisecting CP at right angles is $(2x - x_1)/y_1 + (2y - y_1)/x_1 = 0$, or, from (i.),

$$2(x/b + y/a)(x'/a - y'/b) = a/b + b/a \quad \text{.....(ii.)}.$$

Similarly the line bisecting CP′ at right angles is

$$2(x/b - y/a)(x'/a + y'/b) = a/b + b/a \quad \text{.....(iii.)}.$$

Combining (ii.) and (iii.), the required locus is

$$4(x^2/b^2 - y^2/a^2) = [(a^2+b^2)/ab]^2,$$

i.e., a hyperbola.

Trigonometrical Note: $\tan^{-1}x$ *without imaginaries.*

By R. CHARTRES.

Let $\phi(x) \equiv x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$, convergent $x \not> 1$, and let $\phi(x-h)$, $\phi(x-2h)$, ... be denoted by $\phi_1, \phi_2 \ldots$, h being so small that powers higher than the first may be neglected; then

$$\phi(x) - \phi_1 = h(1 - x^2 + x^4 - \ldots) = h/(1+x^2);$$

but, from the two values of the elementary triangle, $r^2\theta_1 = h$; therefore

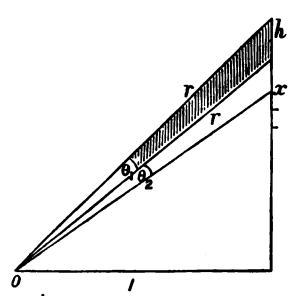
$$\theta_1 = h/(1+x^2);$$

therefore $\theta_1 = \phi(x) - \phi_1$.

Similarly $\theta_2 = \phi_1 - \phi_2$, ...;

therefore

θ or $\tan^{-1}x = \phi(x) \equiv x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$.



15369. (S. C. BAGCHI, B.A.)—ABC is a triangle; a triangle DEF is formed by joining the middle points of AB, BC, CA respectively. FG is the bisector of the angle DFE, meeting DE at G. If AG is perpen-

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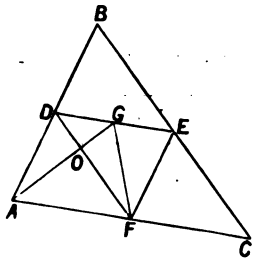
dicular to DF, prove that FG is the diagonal of a square whose side is equal to DG. [Solution by pure geometry preferred.]

Solutions (I.) *by* R. F. Davis, M.A., *and* D. Biddle;
(II.) *by* R. Tucker, M.A.; (III.) *by* A. M. Nesbitt, M.A.

(I.) By supposition,

$$FG^2-DG^2 = AF^2-AD^2;$$



therefore

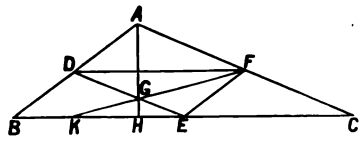
$$DF.EF-DG.EG-DG^2 = DE^2-EF^2$$

or

$$EF(DF+EF) = DE(DE+DG)$$
$$= DE(2DG+EG).$$

But $DF.EF : EF(DF+EF)$
$= DG : DE$ (*Euclid* VI. 3);
hence $DF.EF = DG(2DG+EG)$
and $FG^2 = DF.EF-DG.EG = 2DG^2$.

(II.) $(FG/DG)^2$
$= (\sin C/\sin \frac{1}{2}B)^2 = (2c\cos \frac{1}{2}B/b)^2$
$= c/a.[(a+c)^2-b^2]/b^2$ ..... (i.);
because $\angle DFE$ is bisected,
therefore $DG = ab/[2(a+c)]$.
Because AG is perpendicular to DF, therefore



$$DG = c\cos B/2\cos C;$$

$\therefore$ $2ab\cos C = 2c(a+c)\cos B$,
i.e., $a(a^2+b^2-c^2) = (a+c)(a^2+c^2-b^2)$, *i.e.*, $b^2(2a+c) = c(c+a)^2$... (ii.);
hence from (i.) $(FG/DG)^2 = c/a(2a+c-c)/c = 2$; therefore, &c.

III. Produce ED to M, making DM = DG (so that MB is perpendicular to BC). Draw FL perpendicular to ED.

Now, from the triangle ADF, by *Euclid*, II. 13, we have

$$ED^2 = EF^2+FD^2-2FD.DK$$
$$= EF^2+FD^2-2LD.DG \quad (\because K, F, L, G \text{ are concyclic})$$
$$= EF^2+FG^2-GD^2;$$

therefore $ED^2-DG^2 = GF^2+EF^2-2GD^2$.

But $ED^2-DG^2 = EM.EG$; therefore $EM.EG-EF^2 = GF^2-2GD^2$.

Again $DF.FE = DG.GE+GF^2$;

$\therefore$ $DF.FE-EM.GD = GF^2-DG(EM-GE) = GF^2-2GD^2$,

whence $EM.EG+EM.GD = EF^2+DF.FE$

or $EM.ED = EF(EF+FD)$.

But $ED/(EF+FD) = DG/DF$; therefore $DG/DF = EF/EM$;

therefore $EM.DG = DF.FE = DG.GE+GF^2$;

therefore $DG.GM = GF^2$; therefore $GF^2 = 2GD^2$.

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15275. (R. Tucker, M.A.)—ABC is a triangle, and lines are drawn through the angles, making with the sides angles $= \theta$ and triangles abc, $a'b'c'$. Prove that the circles abc, $a'b'c'$ have for their equations

$$\Delta \Sigma a\beta\gamma = R \sin\theta \, \Sigma a\alpha \, \Sigma c \sin(B-\theta)\,\alpha$$

and

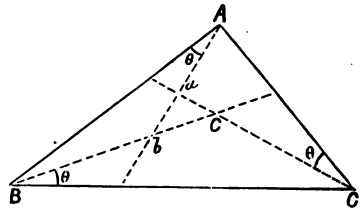
$$\Delta \Sigma a\beta\gamma = R \sin\theta \, \Sigma a\alpha \, \Sigma b \sin(C-\theta)\,\alpha.$$

Hence the radical axis of all the pairs of circles is the circum-Brocardal axis, $\Sigma bc\,(b^2-c^2)\,\alpha = 0$. Find the co-ordinates of the centres of the circles and the equations to the envelopes of the circles.

Solution by the Proposer *and* Professor Sanjána, M.A.

The co-ordinates of a, b, c are respectively

$$\sin(C-\theta)\sin(A-\theta), \quad \sin\theta\sin(A-\theta), \quad \sin^2\theta \quad \ldots\text{(i.)},$$
$$\sin^2\theta, \quad \sin(A-\theta)\sin(B-\theta), \quad \sin\theta\sin(B-\theta) \quad \ldots\text{(ii.)},$$
$$\sin\theta\sin(C-\theta), \quad \sin^2\theta, \quad \sin(B-\theta)\sin(C-\theta) \ldots\text{(iii.)}.$$



From (i.), $\Sigma a\alpha = a\sin^2 A$, $\Sigma a\beta\gamma = a\sin^2 C \sin\theta \sin(A-\theta)$.

Now the determinant corresponding to (i.), (ii.), (iii.) is readily seen to be P^2, where $P \equiv \Pi \sin(A-\theta) - \sin^3\theta = \Pi \sin A \sin(\omega-\theta)/\sin\omega$.

The determinant

$$\sin\theta \begin{vmatrix} c\sin(A-\theta)/a & \sin\theta\sin(A-\theta) & \sin^2\theta \\ a\sin(B-\theta)/b & \sin(A-\theta)\sin(B-\theta) & \sin\theta\sin(B-\theta) \\ b\sin(C-\theta)/c & \sin^2\theta & \sin(B-\theta)\sin(C-\theta) \end{vmatrix}$$

$$= \frac{\sin\theta\sin(B-\theta)}{abc} \begin{vmatrix} bc^2\sin(A-\theta) & \sin\theta\sin(A-\theta) & \sin^2\theta \\ ca^2 & \sin(A-\theta) & \sin\theta \\ ab^2\sin(C-\theta) & \sin^2\theta & \sin(B-\theta)\sin(C-\theta) \end{vmatrix}$$

$$= \frac{c\sin\theta\sin(B-\theta)}{abc} P\,[bc\sin(A-\theta) - a^2\sin\theta]$$

$$= \frac{c\sin\theta\sin(B-\theta)}{abc} P\,(2\Delta\cos\theta - \tfrac{1}{2}\kappa\sin\theta)$$

$$= cP\sin\theta\sin(B-\theta)\,\lambda\sin(\omega-\theta)/abc\,;$$

therefore the equation to the circle abc is

$$\Delta\Sigma a\beta\gamma = R\sin\theta\,\Sigma a\alpha\,\Sigma c\sin(B-\theta)\,\alpha \ldots\ldots\ldots\ldots\text{(iv.)}.$$

This is readily verified.

In a similar manner the equation to the circle $a'b'c'$ is

$$\Delta\Sigma a\beta\gamma = R\sin\theta\,\Sigma a\alpha\,\Sigma b\sin(C-\theta)\,\alpha \ldots\ldots\ldots\ldots\text{(v.)}.$$

Their radical axis is the circum-Brocardal axis of ABC, *i.e.*,

$$\Sigma\, bc\,(b^2-c^2)\,\alpha = 0.$$

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The co-ordinates of the centre of abc are

$$c\sin^2\theta/\sin B + \rho\cos(A-\theta), \quad a\sin^2\theta/\sin C + \rho\cos(B-\theta),$$
$$b\sin^2\theta/\sin A + \rho\cos(C-\theta),$$

where $\rho = R\sin(\omega-\theta)/\sin\omega$; and the co-ordinates of the centre of $a'b'c'$ are

$$b\sin^2\theta/\sin C + \rho\cos(A-\theta), \quad c\sin^2\theta/\sin A + \rho\cos(B-\theta),$$
$$a\sin^2\theta/\sin B + \rho\cos(C-\theta).$$

From (iv.) we have

$$Q = \sin\theta\,[c\sin(B-\theta)\,\alpha + a\sin(C-\theta)\,\beta + b\sin(A-\theta)\,\gamma],$$

where Q stands for $\Delta\,\Sigma a\beta\gamma/(R\,\Sigma aa)$. Then

$$Q = \lambda\sin\theta\cos\theta - \mu\sin^2\theta \quad \text{..... (vi.)},$$

where $\lambda = c\sin B\,\alpha + a\sin C\,\beta + b\sin A\,\gamma$

and $\mu = c\cos B\,\alpha + a\cos C\,\beta + b\cos A\,\gamma$.

Now (vi.) may be written $2Q + \mu = \lambda\sin 2\theta + \mu\cos 2\theta$, $0 = \lambda\cos 2\theta - \mu\sin 2\theta$; hence the equation to the envelope of abc is

$$(2Q+\mu)^2 = \lambda^2 + \mu^2 = c^2a^2 + a^2\beta^2 + b^2\gamma^2 + 2ca\cos(B-C)\,\alpha\beta + 2ab\cos(C-A)\,\beta\gamma + 2bc\cos(A-B)\,\gamma\alpha.$$

In like manner the envelope of $a'b'c'$ may be found. The line of centres is

$$\Sigma a\left[(a^4 - b^2c^2)\sin^2\theta/\sin A + \rho\left\{[a^2 - abc\cos(B-C)]\cos\theta + [4\Delta a - bc\,(b\sin B + c\sin C)]\sin\theta\right\}\right] = 0.$$

15269. (Lt.-Col. Allan Cunningham, R.E.)—(Extension of Question 15147.)—If $X = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}\,.\,X' = X'\Pi\,(p_r^{m_r})$ and $\Sigma\,(p_r, m)$, $\Sigma\,(p_r, n)$ denote the sum of the numbers $\not> X$ containing p_r^m, p_r^n, but not $p_r^{m+\lambda}$, $p_r^{n+\lambda}$, respectively, as factors, show that, when $m+\lambda$, $n+\lambda$ are both $\not>$ the least m_r, then

$$\Sigma\,(\Pi p_r, m) : \Sigma\,(\Pi p_r, n) = \Pi\Sigma\,(p_r, m) : \Pi\Sigma\,(p_r, n).$$

Solution by the Proposer.

Note that X contains both $(p_1 p_2 \dots p_r)^{m+\rho}$ and $(p_1 p_2 \dots p_r)^{n+\rho}$.

Let $S\,(p_r, m)$ = sum of numbers $\not> X$ containing p_r^m.

The actual numbers in $S\,(p_r, m)$ are p_r^m, $2p_r^m$, $3p_r^m$ ... $(X/p_r^m)\,p_r^m$; their *number* being $= X/p_r^m$. Therefore

$$S\,(p_r, m) = p_r^m\,(1 + 2 + 3 + \dots + X/p_r^m) = p_r^m\,(1 + X/p_r^m)\,\tfrac{1}{2}X/p_r^m = \tfrac{1}{2}X\,(1 + X/p_r^m).$$

Similarly, $S\,(p_r, m+\mu) = \tfrac{1}{2}X\,(1 + X/p_r^{m+\rho})$;

therefore $\Sigma\,(p_r, m) = S\,(p_r, m) - S\,(p_r, m+\mu) = \tfrac{1}{2}X\,(X/p_r^m - X/p_r^{m+\rho}) = \tfrac{1}{2}X^2\,[(p_r^\rho - 1)/p_r^{m+\rho}]$.

Similarly, $\Sigma\,(p_r, n) = \tfrac{1}{2}X^2\,[(p_r^\rho - 1)/p_r^{n+\rho}]$;

therefore $\Sigma\,(p_r, m)/\Sigma\,(p_r, n) = p_r^{n-m}$ ..... (A).

Now, write $\Pi p_r = p_1 p_2 \dots p_r$ instead of p_r; therefore

$$\Sigma\,(\Pi p_r, m)/\Sigma\,(\Pi p_r, n) = (p_1 p_2 p_3 \dots p_r)^{n-m} = p_1^{n-m} p_2^{n-m} \dots p_r^{\lambda-m} = \Pi\,(p_r^{n-m}) = \Pi\Sigma\,(p_r, m)/\Pi\Sigma\,(p_r, n),$$

by substitution from result (A). This is the theorem required.

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15353. (H. B. Reynolds.)—In a house are two cisterns respectively 3 ft. cube and 4 ft. cube. It is required to replace these by two new cube cisterns, one less than 3 ft. cube, the other more than 4 ft. cube, so that the two new ones may together contain the same as the two original ones, *i.e.*, 91 cubic feet. The answer must be exact in vulgar fractions —not an approximation in decimals.

Editorial Note.

Mr. R. F. Davis, M.A., draws attention to the interest of Question 15353. Solving by ordinary algebraic methods, he obtained for the edges of the cisterns $1\frac{133843}{338608}$ ft. and $4\frac{102343}{338608}$ ft.; the Proposer, however, by trial, *i.e.*, by the use of a table of cubes, had arrived at far simpler results, viz., (*a*) $\frac{24}{31}$ ft. and $\frac{33}{31}$ ft.; (*b*) $\frac{6543}{1480}$ ft. and $\frac{1457}{1480}$ ft. A further investigation is therefore desirable.

15282. (Hon. G. R. Dick, M.A.)—Prove that, if a is any real quantity, positive or negative,

$$\int_0^\infty \frac{\sin ax}{1+x^2}\,dx = \tfrac{1}{2}e^{-a}\operatorname{li} e^a - \tfrac{1}{2}e^a \operatorname{li} e^{-a},$$

where $\operatorname{li} e^a$ is the logarithmic integral and may be defined by

$$\operatorname{li} e^a = \operatorname{Ei}(a) = \int_0^1 \frac{e^{az}}{z}\,dz - \int_0^\infty \frac{e^{-z\sqrt{a^2}}}{z}\,dz.$$

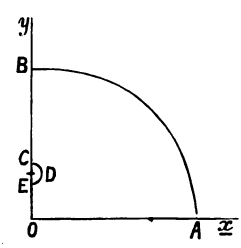
for all real values of a, positive and negative.

[N.B.—The error in the above definition of the li function is noticed in each of the solutions given below.—Editor.]

Solutions (I.) *by* θ. β.; (II.) *by* G. H. Hardy, M.A.; (III.) *by the* Proposer.

(I.) This result is given in Kronecker's *Vorlesungen*, Bd. I. [Vorlesung 12, § 8, equation (17)], with a reference to Schlömilch's *Compendium der höheren Analysis*. Kronecker's solution is the same, except in arrangement, as the following; he adds that Schlömilch's solution is more elementary, but it is not accessible to the present Writer.

Without loss of generality, we may suppose $a > 0$; for, if a is negative, we need only change the sign of the integral to bring it to the former case; and, if $a = 0$, the integral is zero. Then consider the complex integral $\int \frac{e^{iaz}dz}{1+z^2}$ taken round the contour OABCDEO; the arc AB is a quadrant of a circle of large radius R, and CDE is a semicircle of small radius ϵ, whose centre is at the point ($x = 0$, $y = 1$). There is no singularity of the



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integrand within the contour ; and so, by CAUCHY's fundamental theorem, the value of the integral is zero. Writing this out in full, we have

$$\int_0^R \frac{e^{iax}dx}{1+x^2} + \left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^R \frac{e^{-ay}(-idy)}{1-y^2}\right) + \int_{AB} \frac{e^{iaz}dz}{1+z^2} + \int_{CDE} \frac{e^{iaz}dz}{1+z^2} = 0 \ldots (1).$$

It is easily proved that

$$\mathrm{Lt}_{R=\infty} \int_{AB} \frac{e^{iaz}dz}{1+z^2} = 0 \quad (\text{since } a>0),$$

and that

$$\mathrm{Lt}_{\epsilon=0} \int_{CDE} \frac{e^{iaz}dz}{1+z^2} = -\tfrac{1}{2}\pi e^{-a}.$$

Using the last results in equation (1), and taking its real and imaginary parts separately, we have

$$\int_0^\infty \frac{\cos(ax)}{1+x^2}dx = \tfrac{1}{2}\pi e^{-a}, \quad \int_0^\infty \frac{\sin(ax)}{1+x^2}dx = \mathrm{Lt}_{\epsilon=0}\left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^\infty \frac{e^{-ay}dy}{1-y^2}\right).$$

The first result is well known, and is due to LAPLACE ; the second leads to the desired conclusion.

We have

$$\frac{1}{1-y^2} = \tfrac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right),$$

and, since $1+y$ does not vanish for any *positive* value of y, we may write

$$\mathrm{Lt}_{\epsilon=0}\left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^\infty \frac{e^{-ay}}{1-y^2}dy\right) = \tfrac{1}{2}\mathrm{Lt}_{\epsilon=0}\left(\int_0^{1-\epsilon} + \int_{1+\epsilon}^\infty \frac{e^{-ay}}{1-y}dy\right) + \tfrac{1}{2}\int_0^\infty \frac{e^{-ay}}{1+y}dy.$$

In these three integrals we change the variable as follows :—$y = 1-\xi$ in the first, $y = 1+\xi$ in the second, $y = \xi - 1$ in the third. Then

$$\int_0^\infty \frac{\sin(ax)\,dx}{1+x^2} = \tfrac{1}{2}e^{-a}\,\mathrm{Lt}_{\epsilon=0}\left(\int_\epsilon^1 \frac{e^{a\xi}}{\xi}d\xi - \int_\epsilon^\infty \frac{e^{-a\xi}}{\xi}d\xi\right) + \tfrac{1}{2}e^a \int_1^\infty \frac{e^{-a\xi}}{\xi}d\xi.$$

By definition, we have

$$\mathrm{li}\, e^{-a} = -\int_1^\infty \frac{e^{-a\xi}}{\xi}d\xi, \quad \mathrm{li}\, e^a = \mathrm{Lt}_{\epsilon=0}\left(\int_\epsilon^1 \frac{e^{a\xi}}{\xi}d\xi - \int_\epsilon^\infty \frac{e^{-a\xi}}{\xi}d\xi\right) \quad (a>0),$$

and so

$$\int_0^\infty \frac{\sin(ax)\,dx}{1+x^2} = \tfrac{1}{2}e^{-a}(\mathrm{li}\, e^a) - \tfrac{1}{2}e^a(\mathrm{li}\, e^{-a}).$$

It is perhaps advisable to point out that the definition of the li function contained in the question is not entirely exact ; for the two integrals

$$\int_0^1 \frac{e^{a\xi}}{\xi}d\xi, \quad \int_0^\infty \frac{e^{-a\xi}}{\xi}d\xi \quad (a = |a| = +\sqrt{a^2})$$

are not convergent, with zero as the lower limit. The correct definition is, where now a is either positive or negative,

$$\mathrm{li}\, e^a = \mathrm{Lt}_{\epsilon=0}\left(\int_\epsilon^1 \frac{e^{a\xi}}{\xi}d\xi - \int_\epsilon^\infty \frac{e^{-a\xi}}{\xi}d\xi\right) \quad (a = |a|),$$

which takes the forms

$$\text{(i.)}\ \mathrm{li}\, e^a = -\int_1^\infty \frac{e^{-a\xi}}{\xi}d\xi + 2\int_0^1 \frac{\sinh(a\xi)}{\xi}d\xi \quad (a>0),$$

$$\text{(ii.)}\ \mathrm{li}\, e^a = -\int_1^\infty \frac{e^{a\xi}}{\xi}d\xi \quad (a<0).$$

It seems hardly likely that the result of the question can have escaped CAUCHY's notice ; but KRONECKER does not refer to CAUCHY, nor is it given in CAUCHY's classical *Mémoire sur les intégrales définies, prises entre*

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des limites imaginaires (1825), although this memoir contains the evaluation of $\int_0^\infty \frac{\cos(ax)}{1+x^2}\,dx$ by means of a complex integral.

It is possible also to obtain the result of the question by using the formula $$\frac{1}{1+x^2} = \frac{1}{x}\int_0^\infty e^{-y}\sin(xy)\,dy,$$ but then it is necessary to interchange the order of integration, and this is only permissible after certain rather tedious tests [see, for example, STOLZ, *Grundzüge*, Theil III., pp. 31–34, where the corresponding integral with $\cos(ax)$ is evaluated]. Finally we use

$$\int_0^\infty \frac{\cos(\lambda x)-\cos(\mu x)}{x}\,dx = \log\left(\frac{\mu}{\lambda}\right) \quad (\lambda,\ \mu > 0)$$

and integrate by parts. But this method appears longer and less direct than the one depending on CAUCHY'S theorem.

(II.) The expression given in the Question for $\operatorname{li} e^a$ is not correct, as

$$\int^1 \frac{e^{az}}{z}\,dz, \qquad \int_0^\infty \frac{e^{-z\sqrt{a^2}}}{z}\,dz$$

are both divergent. It s! ould be

$$\int_0^1 \frac{e^{az}-e^{-z\sqrt{a^2}}}{z}\,dz - \int_1^\infty \frac{e^{-z\sqrt{a^2}}}{z}\,dz.$$

For this is $$\int_0^a \frac{e^u - e^{-u\operatorname{sgn} a}}{u}\,du + \int_{-\infty}^{-\sqrt{a^2}} \frac{e^u}{u}\,du = \int_{-\infty}^a \frac{e^u}{u}\,du \quad (a < 0),$$

or $$= \lim_{\epsilon=0}\int_\epsilon^a \frac{e^u - e^{-u}}{u}\,du + \int_{-\infty}^{-a}\frac{e^u}{u}\,du$$

$$= \lim_{\epsilon=0}\left(\int_{-a}^{-\epsilon} + \int_\epsilon^a\right)\frac{e^u}{u}\,du + \int_{-\infty}^{-a}\frac{e^u}{u}\,du = \mathrm{P}\int_{-\infty}^a \frac{e^u}{u}\,du \quad (a > 0).$$

These are the definitions usually adopted. The symbol P implies that the principal value of the integral is to be taken.

To solve the Question, integrate $\int \frac{e^{axi}}{1+x^2}\,dx$ $(a > 0)$ round a contour formed by the positive parts of the real and imaginary axes, and a very large quadrant whose centre is O. The point $x = i$ must be avoided by a small semicircle round it. We easily find that

$$\int_0^\infty \frac{\sin ax}{1+x^2}\,dx = \mathrm{P}\int_0^\infty \frac{e^{-ay}}{1-y^2}\,dy = \tfrac{1}{2}\left(\mathrm{P}\int_0^\infty \frac{e^{-ay}}{1-y}\,dy + \mathrm{P}\int_0^\infty \frac{e^{-ay}}{1+y}\,dy\right)$$

$$= \tfrac{1}{2}\left(e^{-a}\mathrm{P}\int_{-\infty}^a \frac{e^u}{u}\,du - e^a\int_\infty^{-a}\frac{e^u}{u}\right)du$$

$$= \tfrac{1}{2}\left(e^{-a}\operatorname{li} e^a - e^a \operatorname{li} e^{-a}\right).$$

(III.) Denoting the value of the integral $\int_0^\infty \frac{\sin ax}{1+x^2}\,dx$ by u, and integrating by parts, we have

$$u = \frac{1}{a} - \frac{2}{a}\int_0^\infty \frac{x\cos ax}{(1+x^2)^2}\,dx \quad \text{or} \quad \frac{1-au}{2} = \int_0^\infty \frac{x\cos ax}{(1+x^2)^2}\,dx.$$

This function is finite and never infinite under the integral sign. Differentiating twice with respect to a, we get

$$\frac{1-au}{2} - \frac{d^2}{da^2}\left(\frac{1-au}{2}\right) = \int_0^\infty \frac{x\cos ax}{1+x^2}\,dx = \frac{du}{da};$$

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or, reducing, $d^2u/da^2 - u = -1/a$. The most general form of solution of this differential equation is $u = \frac{1}{2}e^{-a}\int e^a/a\,da - \frac{1}{2}e^a\int e^{-a}/a\,da$, the arbitrary constants being included under the integral signs. It is clear that it may without loss of generality be exhibited in the form

$$u = \tfrac{1}{2}e^{-a}\{\mathrm{Ei}(a) + C_1\} - \tfrac{1}{2}e^a\{\mathrm{Ei}(-a) + C_2\},$$

where $\mathrm{Ei}(x) = \mathrm{li}(e^x)$ is the exponential integral. From $x = -\infty$ to $x = 0$, $\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^x}{x}\,dx$. From $x = 0$ to $x = +\infty$,

$$\mathrm{Ei}(x) = \mathrm{Ei}(-x) + 2\int_0^1 \frac{\sinh\xi}{\xi}\,d\xi$$

(the definition suggested is incorrect and withdrawn). We have therefore

$$u = \tfrac{1}{2}e^{-a}\,\mathrm{Ei}(a) - \tfrac{1}{2}e^a\,\mathrm{Ei}(-a) + \tfrac{1}{2}C_1e^{-a} - \tfrac{1}{2}C_2e^a.$$

Since u changes sign with a, it is evident that $C_1 = C_2$ ($= 2C$, suppose); thus

$$u = \tfrac{1}{2}e^{-a}\,\mathrm{Ei}(a) - \tfrac{1}{2}e^a\,\mathrm{Ei}(-a) + C(e^{-a} - e^a).$$

We have now to determine C.

Now u is clearly finite ($< \frac{1}{2}\pi$) whatever be the value of a. When a is ∞, $e^{-a}\,\mathrm{Ei}(a)$ takes the form $0 \times \infty$,

$$\frac{\mathrm{Ei}(a)}{e^a} = \frac{\mathrm{Ei}'(a)}{e^a} = \frac{1}{a} = 0$$

when a is ∞. Similarly, $e^a\,\mathrm{Ei}(-a) = 0$ when a is infinite. Thus when a is ∞, if C were not zero, we should have a finite quantity equal to an infinite quantity, since the term Ce^a increases without limit. Thus $C = 0$, and the result is proved.

Some Results in Series.

By Rev. F. H. Jackson.

If $(n)!$ denote

$$(p_1 + p_2 + \ldots + p_n)(p_2 + \ldots + p_n)(p_3 + \ldots + p_n)\ldots(p_{n-1} + p_n)\,p_n;$$

$$(n-1)!,\ (p_1 + p_2 + \ldots + p_{n-1})(p_2 + \ldots + p_{n-1})\ldots(p_{n-2} + p_{n-1})\,p_{n-1};\ \ldots;$$

$$(2)!,\ (p_1 + p_2)\,p_2;\quad (1)!,\ p_1;\quad (0)!,\ 1;\quad \{0\}!,\ 1;\quad \{1\}!,\ p_n;$$

$$\{2\}!,\ p_{n-1}(p_{n-1} + p_n);\quad \{3\}!,\ p_{n-2}(p_{n-2} + p_{n-1})(p_{n-2} + p_{n-1} + p_n);\quad \ldots;$$

$$\{n\}!,\ p_1(p_1 + p_2)(p_1 + p_2 + p_3)\ldots(p_1 + p_2 + \ldots + p_n),$$

in which $p_1, p_2, \ldots, p_n$ are n independent elements, then, for all values of the elements

$$\frac{1}{(n)!\,\{0\}!} - \frac{1}{(n-1)!\,\{1\}!} + \frac{1}{(n-2)!\,\{2\}!} - \ldots + (-1)^n\frac{1}{(0)!\,\{n\}!} \equiv 0 \quad \ldots(1),$$

this is analogous to the series of binomial coefficients

$$\frac{1}{n!\,0!} - \frac{1}{(n-1)!\,1!} + \ldots \equiv 0,$$

to which it plainly reduces if $p_1 = p_2 = p_3 = \ldots = p_n = 1$; for then $(n)! \equiv n!$.

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If the elements are $p_1 = 1$, $p_2 = 2$, ..., $p_n = n$, we also have

$$\frac{1}{(n)!\{0\}!} + \frac{1}{(n-1)!\{1\}!} + \frac{1}{(n-2)!\{2\}!} + \ldots + \frac{1}{(0)!\{n\}!} = \frac{2^n}{n!\,n!} \quad \ldots(2),$$

analogous to $\frac{1}{n!\,0!} + \frac{1}{(n-1)!\,1!} + \ldots = \frac{2^n}{n!}$.

If the elements are $p_1 = 1$, $p_2 = 3$, $p_3 = 5$, ..., $p_n = 2n-1$, the sum of the coefficients is $\frac{2^{2n}}{2n!}$ .....(3);

and in general, if p_1, p_2, ... form an arithmetical progression, $1, 1+a, 1+2a, \ldots 1+(n-1)a$, the sum is

$$\frac{2^n}{n!\,1\,(1+a)\ldots[1+(n-1)a]} \quad \ldots\ldots\ldots\ldots\ldots(4).$$

If the elements are in geometrical progression, viz., $1, p, p^2, \ldots p^{n-1}$, the sum is $2(1+p)(1+p^2)\ldots(1+p^{n-1})$, a well known identity.

An interesting and suggestive form into which (4) may be thrown is

$$(x)_n = (2x)_n + \Sigma \frac{n!}{r!\,n-r!}(2x-2r)_{n-r}(2x-r+1)_r \ldots\ldots\ldots\ldots(5),$$

$(x)_n$ denoting $\frac{1}{x(x-1)\ldots(x-n+1)}$. This again may be thrown into the shape

$$\frac{(x)_n}{(2x)_n} = 1 + n\frac{2x-1}{2x-n} + \frac{n(n-1)}{2!}\frac{2x(2x-3)}{(2x-n)(2x-n-1)} + \ldots \quad \ldots\ldots(6);$$

from which

$$\frac{2n!}{n!\,n!} = 1 + \frac{2n-1}{1!} + \frac{2n(2n-3)}{2!} + \frac{2n(2n-1)(2n-5)}{3!} + \ldots$$
$$+ \frac{2n(2n-1)(2n-2)(2n-7)}{4!} + \ldots \quad \ldots\ldots\ldots\ldots(7),$$

the general term of (6) being $\frac{n!}{r!\,(n-r)!}\frac{(2x-2r)_{n-r}(2x-r+1)_r}{(2x)_n}$.

N.B.—It seems probable that (5), (6), and (7) are capable of extension for all values of n, in which case (6) would be

$$\frac{\pi(-\beta)\pi(-\frac{1}{2}\beta+\alpha)}{\pi(\alpha-\beta)\pi(-\frac{1}{2}\beta)} = 1 + \frac{\alpha(\beta+1)}{1!\,(\beta-\alpha)} + \frac{\alpha(\alpha+1)\beta(\beta+3)}{2!\,(\beta-\alpha)(\beta-\alpha+1)} + \ldots;$$

but the expression on the left is also

$$1 + \frac{\alpha-\beta}{1!\,(2\alpha+2-\beta)} + \frac{\alpha(\alpha+1)\beta(\beta+2)}{2!\,(2\alpha-\beta+2)(2\alpha-\beta+4)} + \ldots$$

by

$$\frac{\pi(\gamma-\alpha-\beta-1)\pi(\gamma-1)}{\pi(\gamma-\alpha-1)\pi(\gamma-\beta-1)} = F(\alpha\beta\gamma).$$

14824. (R. Tucker, M.A.)—ABC is a triangle. AD is the perpendicular from A on BC. A circle, with A as centre and AD as radius, cuts AB, AC in points which are joined. The same is done for the other angles. Prove that these three lines (1) intersect in p, q, r, which lie on the bisectors of A, B, C; (2) they form a triangle similar to that formed by the points of contact of the in-circle with the sides of ABC;

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(3) this in-centre is the orthocentre of pqr; (4) perpendiculars from p, q, r on BC, CA, AB meet in the circum-centre of pqr.

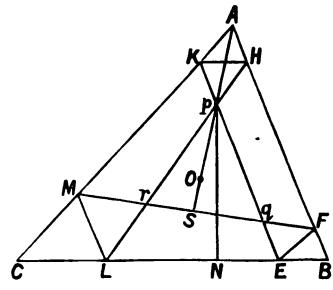
The circum-radius of $pqr = R[\Sigma(\sin A) - 2]$.

Solutions (I.) *by* W. S. Cooney, B.A.; (II.) *by* Lionel E. Reay, B.A.

(I.) Let $AF = AM = AD = \ldots$.

(1) $AH = c - c \sin A$,
$AK = b - b \sin A$;
therefore KH is parallel to BC.

$\angle KHL = \angle BLH = \angle BHL = 90° - \frac{1}{2}\angle B$;
therefore HL bisects $\angle KHB$.
Similarly KE bisects $\angle CKH$;
therefore Ap bisects $\angle A$; therefore &c.



(2) $\angle ApH = \frac{1}{2}\angle AKH = \frac{1}{2}\angle C$;
also $\angle KPA = \frac{1}{2}\angle B$;
therefore $\angle qpr = \frac{1}{2}(\angle B + \angle C) = 90° - \frac{1}{2}\angle A$; therefore &c.

(3) $\angle rpO = \frac{1}{2}\angle C$ and $\angle prq = 90° - \frac{1}{2}\angle C$; therefore OP is perpendicular to qr; therefore &c.

(4) $\angle OpL = \angle ApH = \frac{1}{2}\angle C$ and $\angle NPE = 90° - \angle NEP = \frac{1}{2}\angle C$; therefore $\angle OpL = \angle NpE$; therefore PN passes through the circum-centre of pqr.

Let ρ = circum-radius of pqr.

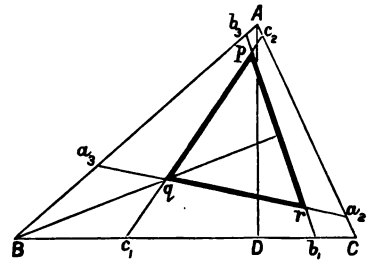
$AS = c \sin B \cos \frac{1}{2}A$, $Ap = [c(1 - \sin A)\cos \frac{1}{2}B] / \sin \frac{1}{2}C$;
therefore

$$ps = \frac{c \cos \frac{1}{2}B (2 \cos \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C - 1 + \sin A)}{\sin \frac{1}{2}C} = \frac{c \cos \frac{1}{2}B (\Sigma \sin A - 2)}{2 \sin \frac{1}{2}C}.$$

$$\rho = \frac{pr}{2 \cos \frac{1}{2}B} = \frac{ps}{2 \cos \frac{1}{2}B \cos \frac{1}{2}C} = \frac{c}{2 \sin C}(\Sigma \sin A - 2) = R(\Sigma \sin A - 2).$$

(II.) Since $Aa_3 = Aa_2$, the bisector of $\angle A$ is perpendicular to the side a_3a_2 or qr. Similarly for the others. Therefore the angle between qp and pr is the supplement of that between the bisectors of $\angle B$ and $\angle C$; therefore

$\angle qpr = \frac{1}{2}(\angle B + \angle C)$.
So $\angle pqr = \frac{1}{2}(\angle A + \angle C)$
and $\angle qrp = \frac{1}{2}(\angle A + \angle B)$;
therefore the triangle qpr is similar to that formed by joining the points of contact of the in-circle of ABC with the sides ..... (2).



Again, if the bisector of $\angle B$ cut a_3a_2 in q_1 and c_1c_2 in q_2,

$Bq_1/Ba_3 = \sin Ba_3q_1 / \sin Bq_1a_3 = \sin(90° + \frac{1}{2}A)/\sin \frac{1}{2}C = \cos \frac{1}{2}A / \sin \frac{1}{2}C$;
and $Ba_3 = c - AD = c(1 - \sin B)$; therefore

$Bq_1 = c(1 - \sin B)\cos \frac{1}{2}A / \sin \frac{1}{2}C = c / \sin C (1 - \sin B) \cos \frac{1}{2}A \cos \frac{1}{2}C$.

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Similarly, $Bq_2 = a/\sin A\,(1-\sin B)\cos\frac{1}{2}A\cos\frac{1}{2}C$; therefore $Bq_1 = Bq_2$; therefore q_1 and q_2 coincide at q; therefore the bisectors of the angles pass through p, q, and r ..... (1).

Since the bisectors of the angles A, B, and C are perpendicular to qr, rp, and pq, and they pass through p, q, and r, therefore their intersection, or the in-centre of ABC, is the orthocentre of pqr ..... (3).

The perpendicular from q on AC makes with qp an angle $= 90° - qc_2C = \frac{1}{2}C$, and the perpendicular from p on BC makes with pq an angle $= \frac{1}{2}C$, and so for the others; therefore these perpendiculars all meet at the circum-centre of pqr ..... (4).

Radius of the circle $pqr = qr/2\sin qpr$. Now

$$qa_3 = Ba_3(\sin\tfrac{1}{2}B/\sin\tfrac{1}{2}C) = 4R\,(1-\sin B)\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C$$

and $$ra_3 = 4R\,(1-\sin C)\sin\tfrac{1}{2}C\cos\tfrac{1}{2}B;$$

also $a_3a_2 = 2Aa_3\sin\frac{1}{2}A = 2AD\sin\frac{1}{2}A = 4R\sin B\sin C\sin\frac{1}{2}A;$

therefore

$$qr = 4R[\sin B\sin C\sin\tfrac{1}{2}A + \sin B\sin\tfrac{1}{2}B\cos\tfrac{1}{2}C + \sin C\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C - \sin\tfrac{1}{2}(B+C)]$$

$$= 4R[\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C(\cos A + \cos B + \cos C - 1 + 2\sin\tfrac{1}{2}B + 2\sin\tfrac{1}{2}C) - \cos\tfrac{1}{2}A]$$

$$= 4R\,(2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C - 1]\cos\tfrac{1}{2}A = 2R[\Sigma(\sin A) - 2]\cos\tfrac{1}{2}A.$$

But $$\sin qpr = \sin\tfrac{1}{2}(B+C) = \cos\tfrac{1}{2}A;$$

therefore circum-radius of $pqr = R[\Sigma(\sin A) - 2]$.

15412. (Professor NANSON.)—If the second differences of a series with an odd number of terms are all positive, show that the arithmetic mean of the odd terms is greater than the arithmetic mean of the even terms.

Solution by **F. W. REEVES, B.A.**

Let $a_1, a_2, a_3, \ldots, a_{2m-1}, a_{2m}, a_{2m+1}$ be the series. Then, since terms of second order of differences are all positive,

$$a_{2m+1} - a_{2m} > a_{2m} - a_{2m-1} > \ldots > a_4 - a_3 > a_3 - a_2 > a_2 - a_1.$$

Assume the proposed theorem true for $2m-1$ terms, *i.e.*,

$$(m-1)(a_1 + a_3 + \ldots + a_{2m-1}) > m(a_2 + a_4 + \ldots - a_{2m-2}) \ldots\ldots (\alpha).$$

Now, if two more terms are added, so that the series consists of $2m+1$ terms, by hypothesis

$$a_{2m+1} - a_{2m} > a_{2m} - a_{2m-1} > a_{2m-2} - a_{2m-3} > \ldots > a_4 - a_3 > a_2 - a_1;$$

therefore $$m(a_{2m+1} - a_{2m}) > (a_2 + a_4 + \ldots + a_{2m}) - (a_1 + a_3 + \ldots + a_{2m-1});$$

therefore $$m \cdot a_{2m+1} + (a_1 + a_3 + \ldots + a_{2m-1}) > (a_2 + a_4 + \ldots + a_{2m}) + m \cdot a_{2m}.$$

Adding this to (α), we have

$$m(a_1 + a_3 + \ldots + a_{2m-1} + a_{2m+1}) > (m+1)(a_2 + a_4 + \ldots + a_{2m});$$

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therefore, if the theorem is true for $2m-1$ terms, it is true for $2m+1$. But, since $a_3-a_2>a_2-a_1$, and therefore $\frac{1}{2}(a_3+a_1)>a_2$, the theorem is true for 3 terms; therefore it is true for 5 terms and therefore for 7 terms, ...; therefore it is universally true that, if the second differences of a series with an odd number of terms are all positive, the arithmetic mean of the odd terms is greater than the arithmetic mean of the even terms.

15395. (Robt. W. D. Christie.) — Let there be two quadratics $Q_1 = x^2+ax+b$, $Q_2 = x^2+cx+d$, and let operative multiplication be performed thus (by Major Ross's method):—$S = [Q_1]\,Q_2$, $\sigma = [Q_2]\,Q_1$, and $s = Q_1Q_2$ by ordinary algebra. Prove that

$$S+\sigma-2s \equiv \{(a-c)^2+a+c\}x^2+\{(b-d)(a-c)+ac\}\,.\,2x + (b-d)^2+b(c+1)+d(a+1)$$

for all integral values of the letters a, b, c, d, x. *E.g.*, $x=2$, $a=11$, $b=13$, $c=3$, $d=5$; then $S=403$, $\sigma=1643$, $s=39\times15$, and consequently $403+1643-1170=876=78\times4+97\times4+176$.

Solution by the Proposer.

We have $S \equiv [x^2+ax+b](x^2+cx+d)$
$\equiv x^4+2cx^3+(a+c^2+2d)x^2+(ac+2cd)x+b+ad+d^2$

and $\sigma \equiv [x^2+cx+d](x^2+ax+b)$
$\equiv x^4+2ax^3+(c+a^2+2b)x^2+(ac+2ab)x+d+bc+b^2$;

also $2s \equiv 2(x^2+ax+b)(x^2+cx+d)$
$\equiv 2x^4+2(a+c)x^3+(ac+b+d)2x^2+2(ad+bc)x+2bd$;

thus $S+s-2s \equiv \{(a-c)^2+a+c\}x^2 + 2\{(a-c)(b-d)+ac\}x+(b-d)^2+b(c+1)+d(a+1)$.

It is clear the principle may be extended.

15396. (C. M. Ross.)—If

$$a^2y^2+b^2x^2=a^2b^2,\quad a^2y(\lambda-x)=b^2x(\mu-y) \text{ ..... (i., ii.)},$$
$$y(a^4y^2+b^4x^2)=a^2b^4(y-\mu) \text{ ..... (iii.)},$$

prove that $a^{\frac{2}{3}}\lambda^{\frac{2}{3}}+b^{\frac{2}{3}}\mu^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

Solutions (I.) *by* S. C. Bagchi, B.A.; (II.) *by* R. Tucker, M.A., *and others*; (III.) *by* H. J. Woodall, A.R.C.S., *the* Proposer, *and others*; (IV.) *by* Rev. T. Wiggins, B.A.

(I.) Let $a^2-b^2=a^2e^2$ $(e<1)$. From (i.) and (iii.) we obtain

$$a^2-e^2x^2=b^2-b^2\mu/y \text{ ..... (A)}.$$

Making use of (ii.),

$$a^2-e^2x^2=a^2(1-\lambda/x) \text{ ..... (iv.)};$$

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therefore $a^2\lambda/x + b^2\mu/y = a^2e^2$ ..... (v.).

Again solving for x and y, we get, from (A) and (iv.), $e^{\frac{1}{2}}x = a^{\frac{1}{2}}\lambda^{\frac{1}{2}}$ and $a^{\frac{1}{2}}e^{\frac{1}{2}}y = b^{\frac{1}{2}}\mu^{\frac{1}{2}}$; therefore, substituting in (v.), $(a\lambda)^{\frac{1}{2}} + (b\mu)^{\frac{1}{2}} = (a^2-b^2)^{\frac{1}{2}}$.

The following is the geometrical interpretation of the above:—Take a fixed straight line ABC, AC being bisected at B. Let (x, y), (λ, μ) be a pair of points $(x > \lambda,\ y > \mu)$ such that, if ϕ be the inclination of the join of these points to AC and if ϕ' be the inclination of (B, x, y) to AC, $\tan\phi : \tan\phi'$ is constant and the difference of the ordinates of the two points bears a constant ratio to the continued product of the greater ordinate and the lines (x, y, A), (x, y, C); then the constants may be so chosen that, while xy is a point on the ellipse whose foci are A and C, λ, μ is on the evolute of the same ellipse.

(II.) Assume $x = a\cos\phi$, $y = b\sin\phi$. These values satisfy (i.). Substituting in (ii.), we get

$$\lambda a\sin\phi - \mu b\cos\phi = (a^2-b^2)\sin\phi\cos\phi \quad \ldots\ldots (\alpha)$$

and in (iii.) $\sin^2\phi(a^2-b^2) = -b\mu$;

hence $\lambda a = (a^2-b^2)\cos^2\phi$ ..... (β);

whence from (α) and (β) we get the result.

(III.) Eliminate y from (i.) and (iii.) by means of (ii.) and consolidate the results; therefore $x^2(a^2-b^2) = a^4\lambda$. In the same way, by elimination of x, $y^2(a^2-b^2) = -b^4\mu$. Substitute these results in (i.) and we get, finally, $a^{\frac{2}{3}}\lambda^{\frac{2}{3}} + b^{\frac{2}{3}}\mu^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$.

(IV.) $y = b\cos\theta$, $x = a\sin\theta$.

$$[a^4y^2 + b^4x^2]/a^2b^2 = [b^2(y-\mu)]/y = [a^2(x-\lambda)]/x = \kappa \quad \text{(suppose)};$$

therefore

$$a^2\cos^2\theta + b^2\sin^2\theta = [b(b\cos\theta-\mu)]/\cos\theta = [a(a\sin\theta-\lambda)]/\sin\theta = \kappa;$$

therefore

$$(a^2-\kappa)\sin\theta = a\lambda, \quad (b^2-\kappa)\cos\theta = b\mu, \quad \text{and} \quad a^2\cos^2\theta + b^2\sin^2\theta = \kappa.$$

This last may be written

$$(a^2-\kappa) = (a^2-b^2)\sin^2\theta \quad \text{or} \quad (b^2-\kappa) = (b^2-a^2)\cos^2\theta;$$

therefore $[(a^2-\kappa)\sin\theta]^{\frac{2}{3}} + [(b^2-\kappa)\cos\theta]^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}(\cos^2\theta + \sin^2\theta)$;

therefore $a^{\frac{2}{3}}\lambda^{\frac{2}{3}} + b^{\frac{2}{3}}\mu^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}$.

15147. (Lt.-Col. Allan Cunningham, R.E.) [Suggested by Question 15126.]—Let $\Sigma(p, a)$ = sum of numbers $\not\equiv X$ containing p^a, but not $p^{a+\lambda}$, where $X = (p^m + q^m)^n - (p^m + q^m)$.

Prove that, when $a+\lambda$, $b+\lambda$ are both $\not\equiv m$,

$$\Sigma(pq, b)/\Sigma(pq, a) = \Sigma(p, b) \cdot \Sigma(q, b)/\Sigma(p, a) \cdot \Sigma(q, a).$$

[All symbols are + integers except p, q, which may be $\pm$.]

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Solution by the PROPOSER.

Since $a+\lambda$, $b+\lambda$ both $\not\equiv m$, therefore X contains $(pq)^{a+\lambda}$ and $(pq)^{b+\lambda}$; the actual numbers $\not\equiv$ X containing p^a are X/p^a in number and are

$$p^a,\ 2p^a,\ 3p^a,\ \ldots,\ (X/p^a)\,.\,p^a.$$

Let $S(p, a)$ denote their sum; then

$$\begin{aligned} S(p, a) &= p^a(1+2+3+\ldots+X/p^a) \\ &= p^a(1+X/p^a)\,.\,\tfrac{1}{2}X/p^a = \tfrac{1}{2}X(1+X/p^a), \\ S(p, a+\lambda) &= \tfrac{1}{2}X(1+X/p^{a+\lambda}); \end{aligned}$$

therefore $\Sigma(p, a) = S(p, a) - S(p, a+\lambda)$

$$= \tfrac{1}{2}X(X/p^a - X/p^{a+\lambda}) = (p^\lambda - 1)/(p^{a+\lambda})\,.\,\tfrac{1}{2}X^2.$$

$\Sigma(p, b) = (p^\lambda-1)/p^{b+\lambda}\,.\,\tfrac{1}{2}X^2$; therefore $\Sigma(p, b)/\Sigma(p, a) = p^{a-b}$ and $\Sigma(q, b)/\Sigma(q, a) = q^{a-b}$. Similarly $\Sigma(pq, b)/\Sigma(pq, a) = (pq)^{a-b}$. This proves the proposition.

15383. (A. M. NESBITT, M.A.)—If $\alpha+\beta+\gamma$ be a multiple of π, and if $x^r \sin r\alpha + y^r \sin r\beta + z^r \sin r\gamma$ vanishes when $r = 1$ and when $r = 2$, it vanishes for any positive integral value of r.

Solution by Professor SANJÁNA, M.A.

From $x\sin\alpha + y\sin\beta = -z\sin\gamma$ and $x^2\sin 2\alpha + y^2\sin 2\beta = -z^2\sin 2\gamma$ we deduce $x^2\sin 2\alpha + y^2\sin 2\beta = -2\cot\gamma\,(x\sin\alpha + y\sin\beta)^2$, or, on rearranging, $x^2+y^2+2xy\cos\gamma = 0$, also two similar results. Hence

$$\begin{aligned} xy\sin\gamma + yz\sin\alpha + zx\sin\beta &= -\tfrac{1}{2}(x^2+y^2)\tan\gamma - \ldots \\ &= -\tfrac{1}{2}x^2(\tan\gamma+\tan\beta) - \ldots = -\tfrac{1}{2}x^2\sin\alpha/\cos\beta\cos\gamma - \ldots \\ &= -(1/4\cos\alpha\cos\beta\cos\gamma)(x^2\sin 2\alpha + y^2\sin 2\beta + z^2\sin 2\gamma) = 0. \end{aligned}$$

Now $(1+xe^{\alpha_1})(1+ye^{\beta_1})(1+ze^{\gamma_1})$

$$\begin{aligned} &= 1 + (xe^{\alpha_1} + ye^{\beta_1} + ze^{\gamma_1}) + (xy^{k_1}e^{\pi-\eta} + \ldots) + xyze^{i(\alpha+\beta+\gamma)} \\ &= 1 + \Sigma xe^{\alpha_1} \pm \Sigma xye^{-\gamma_1} \pm xyz; \end{aligned}$$

and $(1+xe^{-\alpha_1})(1+ye^{-\beta_1})(1+ze^{-\gamma_1}) = 1+\Sigma xe^{-\alpha_1} \pm \Sigma xye^{\gamma_1} \pm xyz.$

As $\Sigma x(e^{\alpha_1} - e^{-\alpha_1}) = 2i\Sigma x\sin\alpha$ and $\Sigma xy(e^{\gamma_1} - e^{-\gamma_1}) = 2i\Sigma xy\sin\gamma$,

these two expressions are equal. Take their logarithms and equate the coefficients of x^r; we get

$$[(-1)^{r-1}/r](x^re^{r\alpha_1} + y^re^{r\beta_1} + z^re^{r\gamma_1}) = [(-1)^{r-1}/r](x^re^{-r\alpha_1} + y^re^{-r\beta_1} + z^re^{-r\gamma_1}),$$

i.e., $2i\Sigma x^r\sin r\alpha = 0$; whence the proposition.

15422. (A. ENSOR.)—P and Q are taken at random on a given line AB. Find the mean of

(1) AP . PQ . QB, (2) ΣAP . PQ, (3) $AP^2 + PQ^2 + QB^2$.

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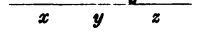
Solution by R. CHARTRES.

$AB^3 \equiv 1^3 = (x+y+z)^3$ containing 10 terms; therefore $M(xyz) = \frac{3!}{3^3}$. $AB^2 \equiv 1^2 = (x+y+z)^2$ containing 6 terms; therefore



$$M(2xy) = 3M(xy) = \tfrac{1}{4}$$

and $$M(x^2+y^2+z^2) = 3M(x^2) = \tfrac{1}{3},$$

by Rev. Prebendary WHITWORTH'S theorem.

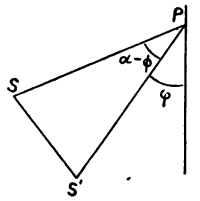
9081. (Professor WOLSTENHOLME, M.A., Sc.D.)—At each point P of a given equiangular spiral is drawn a curve similar to the curve $r^n = a^n \sin n\theta$, having four-point contact with the spiral; S is the pole of the spiral, S′ that of the osculating curve; prove that $SS' = n \cdot SP$, and that the angle PSS′ is double the angle between S′P and the tangent at P.

[Thus, if the osculating curve be a parabola or cardioide, $SS' = \frac{1}{2}SP$; if it be a rectangular hyperbola or lemniscate of BERNOULLI, $SS' = 2SP$. The theorem is not true if $n = \pm 1$, four-point contact being then impossible.

Solution by FRANCES E. CAVE.

For the spiral,

$$\rho = \frac{SP}{\sin\alpha}, \qquad \frac{d\rho}{d\psi} = \rho\cot\alpha.$$



For the other curve,

$$\rho = \frac{S'P}{(n+1)\sin\phi}, \quad \psi = (n+1)\,\theta,$$

$$\frac{d\rho}{d\psi} = -\frac{n-1}{(n+1)^2} S'P\cot\phi\,\operatorname{cosec}\phi = -\frac{n-1}{n+1}\rho\cot\phi;$$

therefore $(n+1)\cot\alpha = (1-n)\cot\phi$;

and therefore $$n\sin(\alpha+\phi) = \sin(\alpha-\phi),$$

$$\frac{SP}{S'P} = \frac{\sin\alpha}{(n+1)\sin\phi} = \frac{\sin(\alpha+\phi)}{\sin 2\phi},$$

and hence $$\angle PSS' = 2\phi, \quad \angle SS'P = \pi - (\alpha+\phi),$$

and $$\frac{SS'}{SP} = \frac{\sin(\alpha-\phi)}{\sin(\alpha+\phi)} = n.$$

15286. (R. W. D. CHRISTIE.) — Extend EULER'S theorem of two squares to the cubes: *e.g.*, $(1^3+19^3)(17^3+37^3) = 683^3+397^3$.

Note by JAMES BLAIKIE, M.A.

EULER'S theorem of two squares shows that, for all values of a, b, c, d, $(a^2+b^2)(c^2+d^2)$ is equal to the sum of two squares *in two ways*, viz., $(ac+bd)^2+(ad-bc)^2$ and $(ac-bd)^2+(ad+bc)^2$. It is to be observed that when the theorem is extended the cubes a, b, c, d must satisfy a certain condition and that there is only one way in which the sum of two cubes can be obtained. Let

$$(a^3+b^3)(c^3+d^3) = (ac+bd-x)^3+(ad+bc+x)^3 = M^3+N^3.$$

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This reduces to the quadratic equation

$$x^2-(ac+bd-ad-bc)\,x+abcd = x^2-2Ax+B = 0.$$

For rational values of x we must therefore have A^2-B a square number; that is, $a^2c^2+a^2d^2+b^2c^2+b^2d^2-2a^2cd-2b^2cd-2c^2ab-2d^2ab$ is a square number. Also, if x_1, x_2 be the values of x, and M_1, N_1, M_2, N_2 the corresponding values of M and N, we have $x_1+x_2 = 2A = ac+bd-ad-bc$; therefore $ac+bd-x_1 = ad+bc+x_2$ and $ac+bd-x_2 = ad+bc+x_1$; or $M_1 = N_2$, $M_2 = N_1$. There is, therefore, only one solution.

Example.—$a = 1$, $b = 19$, $c = 17$, $d = 37$ gives $x = 37$ or 323; therefore $M = 683$, $N = 397$; therefore $(1^3+19^3)(17^3+37^3) = (683)^3+(397)^3$.

Other cases may be easily found by making use of the relation

$$x_1x_2 = abcd.$$

Additional Note by the PROPOSER.

Mr. BLAIKIE's note is valuable in itself, but no method of getting other integral values is given from the equation necessary by his method $A^2-B = \square$ or $(ac+bd-ad-bc)^2-4abcd = \square$.

The following gives a very simple solution of the problem:—We have

$$(n^2+2n-2)^3+(2n^2-2n-1)^3 \equiv \{(2n-1)^3+(n-2)^3\}\{n^3+1^3\}.$$

Many other similar identities can be formed to meet requirements from

$$c/d = (2a-b)/(a-2b), \text{ when } (a^3+b^3)(c^3+d^3) = X^3+Y^3.$$

NOTE.—We may also put $(mn)^3+m^3$ in place of n^3+1^3.

15171. (A. M. NESBITT, M.A.)—Discuss the curves—

(1) $(y-x^2)^2 = xy^4$, (2) $(y-x^2)^2 = x(x^4-y^4)$.

Solution by the PROPOSER.

(1) Curve $(y-x^2)^2 = xy^4$.—By putting $y = mx^2$, we get a series of points on the curve given by $x = (m-1)^{\frac{1}{2}}/m^{\frac{1}{2}}$,

$y = (m-1)^{\frac{1}{2}}/m^{\frac{1}{2}}$;

and, similarly, by putting $y = -mx^2$, we get

$x = (m+1)^{\frac{1}{2}}/m^{\frac{1}{2}}$,

$y = -(m+1)^{\frac{1}{2}}/m^{\frac{1}{2}}$.

The equation also yields

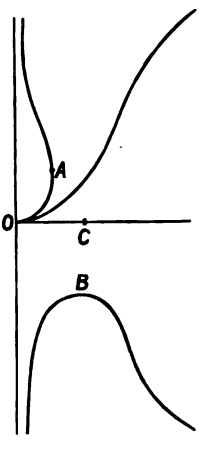
$y^2 = \pm(y/x^{\frac{1}{2}}-x^{\frac{5}{2}})$,

which for the upper sign gives

$y = -1/2x^{\frac{1}{2}} \pm \sqrt{(1/4x+x^{\frac{1}{2}})}$...(1),

and for the lower sign

$y = 1/2x^{\frac{1}{2}} \pm \sqrt{(1/4x-x^{\frac{1}{2}})}$ ... (2).



A is (·57, ·66);
B is (·72, −1·19);
OC = 1.

Taking the upper sign in (1) or lower sign in (2), we see that for x small both values of y are positive; hence there is a cusp of the second species at the origin. Taking the lower sign in (1) or upper

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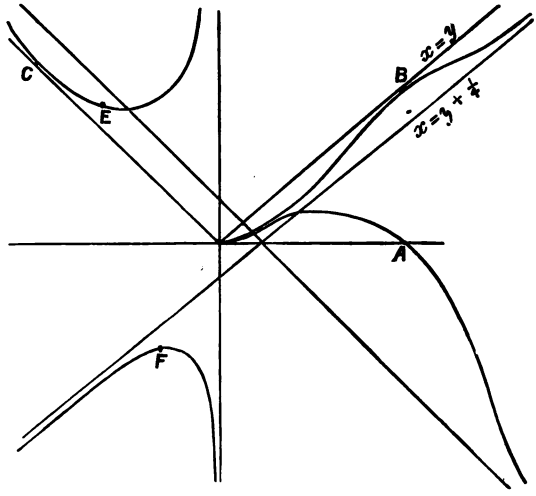
sign in (2), the two infinite values of y for x small are of opposite sign; hence there is a point of inflexion at infinity on the asymptote $x = 0$.

Again, $u_x = 4x(x^2-y)-y^4$, $u_y = 2(y-x^2)-4xy^3$ ..... (3, 4). Solving (3) with the curve, we obtain $y = -3x^2$, which gives $x = 2^{\frac{1}{3}}/3^{\frac{1}{3}}$, $y = -2^{\frac{2}{3}}/3^{\frac{2}{3}}$, or, approximately, (·72, −1·19), as the point where the curve is parallel to the axis of x. Solving (4) with the curve, we get $y - 1/16y^4 = \pm\frac{7}{8}$ or $y^5 = \frac{1}{8}$ or $\frac{1}{24}$, the corresponding value for x^5 being $\frac{1}{16}$ or $\frac{9}{16}$. When $x > 2^{-\frac{1}{3}}$ the two values of y given by (2) become impossible; the other point ($x^5 = \frac{9}{16}$, $y^5 = \frac{1}{24}$) does not, in fact, satisfy (4), since it gives $y - x^2 = -2xy^3$. The approximate co-ordinates of the point for which $u_y = 0$ are (·57, ·66). The ultimate form of the curve is given by $y^4 = x^3$. Hence the figure is as appended.

(2) Curve $(y-x^2)^2 = x(x^4-y^4)$.—The asymptotes are $x = 0$, $x \pm y = \frac{1}{2}$. The points A (1, 0), B (1, 1), C (−1, 1) are on the curve; and the corresponding values of dy/dx will be found to be $-\frac{1}{2}$, 1, −1. If $y^2 > x^2$, x is negative, and not otherwise. There is a cusp of the second species



at the origin. There are practical difficulties in determining the precise positions of E and F where the curve is parallel to the x axis; but the general form of the curve is as shown. It is easily established that the cubic to determine the abscissæ of the points where the curve cuts the asymptote $x - y = \frac{1}{4}$ has no negative root.

15415. (Professor NEUBERG.)—Une parallèle au côté BC d'un triangle ABC coupe les côtés AB, AC en B′, C′. Comment faut-il mener cette droite pour que les volumes engendrés par le triangle AB′C′ et par le trapèze B′C′CB tournant autour de B′C′ soient dans un rapport donné r?

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Solution by H. J. Woodall, A.R.C.S.

The centre of gravity of AB′C′ is distant from B′C′ by $\frac{1}{3}$AB′ sin B, that of B′C′CB distant from B′C′ by

$$\tfrac{1}{3}(AB-AB')\sin B\,\frac{AB'+2AB}{AB'+AB}.$$

Write $AB'/AB = x$;

first volume $= \frac{1}{3}x\,AB\sin B\,.\,2\pi\,.\,\frac{1}{3}x^2\,AB\sin B\,.\,BC = \frac{1}{3}\pi x^3 AB^2\,.\,BC\sin^2 B$,

second volume

$= [\frac{1}{3}(AB^3-AB'^3)/(AB^2-AB'^2)-AB']\sin B\,.\,2\pi\,.\,\frac{1}{3}AB\,.\,BC\sin B\,(1-x^2)$
$= \pi AB^2\,.\,BC\,(1-x^2)\sin^2 B\,[(2-3x+x^3)/3\,(1-x^2)]$;

therefore $r = \pi x^3/\{\pi(2-3x+x^3)\} = x^3/(2-3x+x^3)$.

14793. (Professor A. Droz-Farny.)—Démontrer que dans un triangle on a toujours $p^2-bc > S\sqrt{3}$.

Solution by C. E. Youngman, M.A.

Put in the usual English notation, this is $s^2-bc > \Delta\sqrt{3}$. Now the expression

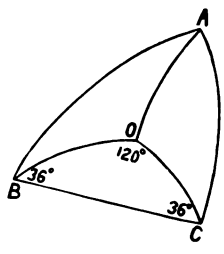
$X[\equiv(s^2-bc)/\Delta] = s/r-bc/\Delta = \Sigma(s-a)/r-2\operatorname{cosec}A = \Sigma\cot\frac{1}{2}A-2\operatorname{cosec}A$
$= \cot\frac{1}{2}B+\cot\frac{1}{2}C-\tan\frac{1}{2}A = \cot\frac{1}{2}(\pi+A)+\cot\frac{1}{2}B+\cot\frac{1}{2}C$.

Here the three angles are those of a triangle (of IBC in fact, or ABI^b, or AI$_c$C); therefore the minimum of X is the same as that of $\Sigma\cot A$, namely, $\sqrt{3}$ (obtained when A = B = C).

15218. (James Blaikie, M.A.)—If x be the edge of an icosahedron and y the radius of the circum-sphere, show that $5x^4-20x^2y^2+16y^4 = 0$.

Solutions (I.) *by* Professor Sanjána, M.A.; (II.) *by the* Proposer *and* B. C. Wallis, F.C.P.; (III.) *by* I. Arnold.

(I.) As there are five plane angles in any solid angle, the angle BAC (formed by passing planes through the centre and two adjacent edges) = 72°. If O be the pole of the circum-circle of the triangle BAC, ∠ BOC = 120°, ∠ OBC = ∠ OCB = 36°. Hence



$\cos BC = (\cos 120° + \cos^2 36°)/\sin^2 36° = 1/\sqrt{5}$.

Now $x = 2y\sin\frac{1}{2}BC$; therefore

$x^2 = 2y^2(1-\cos BC = 2y^2-(2/\sqrt{5})\,y^2$;

therefore $x^4-4y^2x^2+4y^4 = \frac{4}{5}y^4$,

i.e., $5x^4-20x^2y^2+16y^4 = 0$.

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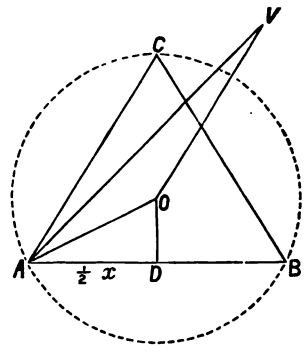
(II.) $y = \frac{1}{2}x \tan \frac{1}{2}\mathrm{I} \tan 36°$, where $\sin \frac{1}{2}\mathrm{I} = \cos 36°/\sin 60°$. Substituting the values, we get

$$4y^2 = x^2 \tan^2 \tfrac{1}{2}\mathrm{I} \tan^2 36° = x^2 . \tfrac{1}{2}(7 + 3\sqrt{5})(5 - 2\sqrt{5}) = \tfrac{1}{2}x^2(5 + \sqrt{5});$$

therefore $\sqrt{5}\, x^2 = 8y^2 - 5x^2$, which, on squaring and reduction, gives the required relation.

(III.) Let ABC be the equilateral triangle of the icosahedron, O the centre of the circum-circle, V the centre of the sphere, VO the perpendicular on the triangle, and AO the radius of the circle. Then



$AO = x/\sqrt{3}$ and $y^2 - \frac{1}{3}x^2 = VO^2$;

therefore $VO = \sqrt{(y^2 - \frac{1}{3}x^2)}$

and $\frac{1}{4}(x^2\sqrt{3})\,\frac{1}{3}\sqrt{(y^2 - \frac{1}{3}x^2)}$

$= $ vol. of triangular pyramid

$= \frac{1}{12}(x^2\sqrt{3})\sqrt{(y^2 - \frac{1}{3}x^2)}$;

and multiplied by 20

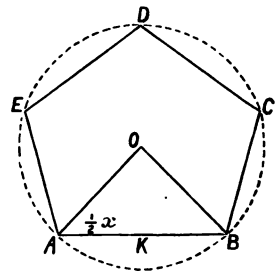
$= \frac{1}{3}(5x^2\sqrt{3})\sqrt{(y^2 - \frac{1}{3}x^2)}$

$= $ vol. of icosahedron;

and squared $= \frac{75}{9}x^4(y^2 - \frac{1}{3}x^2) = \frac{25}{3}x^4(y^2 - \frac{1}{3}x^2) = \frac{25}{3}x^4y^2 - \frac{25}{9}x^6$

$= \frac{1}{9}(75x^4y^2 - 25x^6) = $ vol.^{2} of icosahedron ..... (1.)

Next, let ABCDE be the pentagon of the icosahedron, and $AK = \frac{1}{2}x$.



$\cos 54° : \sin 54° = \frac{1}{2}x : OK$,

or $\sqrt{(10 - 2\sqrt{5})} : (1 + \sqrt{5}) = \frac{1}{2}x : OK$;

therefore $OK = \dfrac{x(1 + \sqrt{5})}{2\sqrt{(10 - 2\sqrt{5})}}$ and

$\dfrac{x^2(1 + \sqrt{5})}{4\sqrt{(10 - 2\sqrt{5})}} = $ area of $\triangle AOB$;

and multiplied by 5

$= $ area of pentagon $= \dfrac{5x^2(1 + \sqrt{5})}{4\sqrt{(10 - 2\sqrt{5})}}$;

and multiplied by $\frac{1}{3}y = \dfrac{20x^2y(1 + \sqrt{5})}{12\sqrt{(10 - 2\sqrt{5})}} = $ vol. of icosahedron

$$= \tfrac{5}{3}x^2y \frac{(1 + \sqrt{5})}{\sqrt{(10 - 2\sqrt{5})}} = \tfrac{5}{3}x^2y \frac{-4}{\sqrt{(80 - 32\sqrt{5})}}$$

$$= \frac{-20x^2y}{12\sqrt{(5 - 2\sqrt{5})}} = \frac{-5x^2y}{3\sqrt{(5 - 2\sqrt{5})}};$$

and this squared $= \dfrac{25x^4y^2}{9(5 - 2\sqrt{5})} = \dfrac{25x^4y^2}{45 - 18\sqrt{5}}$ .....(2);

therefore, from (1) and (2), by subtraction,

$$\tfrac{1}{9}(75x^4y^2 - 25x^6) = \frac{25x^4y^2}{45 - 18\sqrt{5}} \text{ and } \frac{25y^2}{45 - 18\sqrt{5}} = \tfrac{1}{9}(75y^2 - 25x^2);$$

and, clearing of fractions,

$$225y^2 = 3375y^2 - 1125x^2 - 1350\sqrt{5}\, y^2 + 450\sqrt{5}\, x^2.$$

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Dividing and simplifying, we get $6\sqrt{5}y^2 - 2\sqrt{5}x^2 = 14y^2 - 5x^2$; and this squared gives

$180y^4 - 120x^2y^2 + 20x^4 = 196y^2 - 140x^2y^2 + 25x^4$, or $5x^4 - 20x^2y^2 + 16y^4 = 0$, the equation in question.

[The rule for finding the volume of the icosahedron—viz.: multiplying the area of the pentagon by $\frac{1}{3}$ of the radius of the circumscribing sphere—was originally proposed by me in the *Ed. Times* for 1889, or thereabouts.]

15373. (R. W. D. Christie.)—For all primes of form $4m+1$, prove $r^{6m} - r^{4m} + 2 \equiv 0 \pmod{p}$ (r a prime root), and give an example of its utility for factorization. *E.g.*, $12053^6 - 12053^4 + 2 \equiv 0 \pmod{12301}$.

Solutions (I.) *by* Lt.-Col. Allan Cunningham, R.E.;
(II.) *by the* Proposer.

I. Here (r being a primitive root of p) $r^{4m} \equiv +1$, $r^{2m} \equiv -1 \pmod{p}$; therefore

$$r^{6m} - r^{4m} \equiv -1 - 1 \equiv -2 \pmod{p},$$

which proves the theorem. This can be used in factorization as follows. It gives

$$\{(np \pm r)^{6m} - (np \pm r)^{4m} + 2\} \equiv 0 \pmod{p};$$

so that the quantity inside the { } contains p.

Or, again, find ρ such that $\rho^2 \equiv -1 \pmod{p}$; this gives $r^{2m} \equiv \rho^2$, $r^m \equiv \pm\rho \pmod{p}$; hence

$$\{\rho^6 - \rho^4 + 2\} \equiv 0 \quad \text{and} \quad \{(np \pm \rho)^6 - (np \pm \rho)^4 + 2\} \equiv 0 \pmod{p};$$

so that the quantity within the { } contains p.

Ex.—It will be found that $248^2 \equiv -1 \pmod{12301}$. Here $p - \rho = 12053$; therefore $(12053^6 - 12053^4 + 2)$ contains 12301.

II. Where p_{4m+1} means any prime of form $4m+1$, by the theorem of primitive roots, we have $r^{2m} + 1 = 0 \pmod{p_{4m+1}}$ (r any prime root); also $2(r^{6m} + 1) = 0$ and $r^{4m}(r^{2m} + 1) = 0$, whence $r^{6m} - r^{4m} + 2 = 0$.

Similarly,

$$r^{6m} - nr^{4m} + (n+1) = 0 \quad (n \text{ arbitrary}),$$

or

$$(2a+1)^6 - n(2a+1)^4 + n + 1 = 0 \pmod{2a^2 + 2a + 1};$$

and many others by combination.

N.B.—Since $r^{6m} - r^{4m} + 2 \equiv (r^m)^6 - (r^m)^4 + 2$, we have only to secure r^m (for which there is a mechanical process for any prime of given form, however large, independent of r). Here (see Question 15354) $248 \times 249 = 247 \pmod{12301}$ or $247 \times 248 = -249 \pmod{12301}$, and we have

$$248^6 - 248^4 + 2 = 12053^6 - 12053^4 + 2 = 0 \bmod 12301.$$

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15388. (H. Bateman.)—S, S′, and C are the foci and centre of a fixed conic touching four lines; T, T′, and O the foci and centre of a variable conic touching the four lines. Prove that ST.ST′, S′T.S′T′, and CO are to one another in constant ratios.

Solutions (I.) *by the* Proposer; (II.) *by* Rev. R. Monteith.

(I.) Consider a system of curves of the n-th class touching n^2 straight lines. If p_1, p_2, p_3 are the distances of three points (x_1, y_1), (x_2, y_2), (x_3, y_3) from a straight line, the tangential equation will be of the form $F(p_1p_2p_3)+\lambda\Phi(p_1p_2p_3)=0$ where F and Φ are homogeneous algebraic functions of the n-th degree. Let the point (x, y) be a focus of this curve; then the line $(\xi-x)+i(\eta-y)=0$ is a tangent. Writing $z=x+iy$, the foci are given by $0=F[z_1-z, z_2-z, z_3-z]+\lambda\Phi[z_1-z, z_2-z, z_3-z]$. Let $c_1, c_2, \ldots, c_n$ be the points given by the n roots of the equation $F[z_1-z, \ldots]+\lambda\Phi[z_1-z, \ldots, z_3-z]=0$; these will be the real foci of the curve. Let $A_1 \ldots A_n$, $B_1 \ldots B_n$ be the corresponding foci for the curves $\lambda=0$, $\lambda=\infty$; then

$$P(z-a_1)\ldots(z-a_n)+Q\lambda(z-b_1)\ldots(z-b_n)=(P+Q\lambda)(z-c_1)\ldots(z-c_n) \quad (1)$$

where P, Q, and λ are real; therefore

$$(b_1-a_1)\ldots(b_1-a_n)=(1+Q/P\lambda)(b_1-c_1)\ldots(b_1-c_n).$$

From this equation we may draw two inferences:

(1) The sum of the angles made by $B_1A_1, \ldots, B_1A_n$ with any line equals the sum of the angles made by $B_1C_1, \ldots, B_1C_n$ with that line.

(2) $(B_1A_1.B_1A_2\ldots B_1A_n)/(B_1C_1.B_1C_2\ldots B_1C_n)$
$$=1+Q/P\lambda=(B_2A_1.B_2A_2\ldots B_2A_n)/(B_2C_1.B_2C_2\ldots B_2C_n).$$

Equating coefficients of z in equation (1), we have

$$P(a_1+a_2+\ldots+a_n)+Q\lambda(b_1+\ldots+b_n)=(P+Q\lambda)(c_1+\ldots+c_n).$$

Now the point $z=(c_1+c_2+\ldots+c_n)/n$ is the centre of gravity of the real foci of the curve λ, *i.e.* the centre of the curve. Denoting this point by C and the corresponding points by A and B, the above equation shows that A, B, C are collinear and that $BA/BC=1+Q/P\lambda$; therefore

$$(B_1A_1.B_1A_2\ldots B_1A_n)/(B_1C_1.B_1C_2\ldots B_1C_n)=\ldots=BA/BC.$$

Taking $n=2$, we have the theorem in the Question.

(II.) Taking S or S′ as origin, we can write the fixed conic

$$K_1(l^2+m^2)+(2l\gamma-1)=0$$

and the variable conic (λ) in either of the forms

$$K_1(l^2+m^2)-\lambda(la+mb-1)(la_1+mb_1-1)+2l\gamma-1=0 \quad \ldots\ldots\ldots(1),$$

$$K(l^2+m^2)-(1+\lambda)(la+m\beta-1)(la_1+m\beta_1-1)=0 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

where (ab), (a_1b_1) are fixed meets of common tangents and $(\alpha\beta)$, $(\alpha_1\beta_1)$ are T, T′. Therefore

$$(K-K_1)(l^2+m^2)+l^2[aa_1\lambda-aa_1(1+\lambda)]+lm[(ab_1+a_1b)\lambda+(a\beta_1+a_1\beta)(1+\lambda)]$$
$$+m^2(\ldots)+l[(a+a_1)\lambda-(a+a_1)(1+\lambda)+2\gamma]$$
$$+m[(b+b_1)\lambda-(1+\lambda)(\beta+\beta_1)]\equiv 0;$$

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therefore $\frac{\lambda}{1+\lambda} = \frac{\beta+\beta_1}{b+b_1} = \frac{a\beta_1+a_1\beta}{ab_1+a_1b} = \frac{aa_1-\beta\beta_1}{aa_1-bb_1} = \frac{a+a_1-2\gamma}{a+a_1-2\gamma}$;

therefore $ST^2.ST'^2 = [\lambda/(1+\lambda)]^2[(ax_1-bb_1)^2+(ab_1+a_1b)^2]$,

$CO^2 = [\lambda/(1+\lambda)]^2\{[\frac{1}{2}(a+a_1)-\gamma]^2+[\frac{1}{2}(b+b_1)]^2\}$;

therefore $(ST.ST')/CO$ is independent of λ.

15319. (J. L. S. Hatton.) (*Cf.* Question 12100, Thos. Muir, LL.D., F.R.S.E.)—Solve the equation $x^4+px^2+qx+r=0$ by comparing the coefficients with those in the circulant

$$\begin{vmatrix} x & a & b & c \\ c & x & a & b \\ b & c & x & a \\ a & b & c & x \end{vmatrix}$$

equated to zero.

Solution by A. M. Nesbitt, M.A.

We must take

$p=-2(b^2+2ac)$, $q=4b(a^2+c^2)$, $r=b^2(b^2-4ac)-(a^2-c^2)^2$ ... (1, 2, 3),

and b^2 will be given by the cubic

$$64b^6+32pb^4+4b^2(p^2-4r)-q^2=0.$$

Two values of x are then $-b\pm(a+c)$, as is obvious from the determinant; while $(a+c)^2=q/4b-\frac{1}{2}p-b^2$. Thus we get $x=-b\pm\sqrt{(q/4b-\frac{1}{2}p-b^2)}$, and, as b is susceptible of a double sign, we get four values for x from each for b^2. It is easy to see that in a particular case these three sets of four values are mere repetitions: take, *e.g.*, the equation $x^4-25x^2+60x-36=0$. The three values of b^2 are 4, $\frac{9}{4}$, $\frac{25}{4}$; and the values for x come out as follows: — (i.) $(\pm4-2)$, $(\pm1+2)$; (ii.) $\frac{1}{2}(\pm9-3)$, $\frac{1}{2}(\pm1+3)$; (iii.) $\frac{1}{2}(\pm7-5)$, $\frac{1}{2}(\pm1+5)$. The cubic written down above is in fact identical with the auxiliary cubic in Euler's solution of a biquadratic. The present method is, however, superior on account of its not producing any inapplicable results, as Euler's does.

[It may be noticed that, if two or four roots of the biquadratic are imaginary, only one value of b^2 will give these results in a presentable form. For, if $\alpha, \beta, \gamma, \delta$ denote the roots, the three values of b^2 give them to us linked in the following ways: — $(\alpha, \beta; \gamma, \delta)$, $(\alpha, \gamma; \beta, \delta)$, and $(\alpha, \delta; \beta, \gamma)$; and, unless the conjugate roots are associated, the results will be practically useless. Thus, in the equation $x^4+2x^2+4x+8=0$ we find $b^2=1$ or $-1\pm\frac{1}{2}\sqrt{3}$. The first value gives $-1\pm\sqrt{(-1)}$ and $1\pm\sqrt{(-3)}$; but, if we take $b^2=-\frac{1}{2}(2+\sqrt{3})$, so that

$$b=\pm\tfrac{1}{2}\sqrt{(-1)}(\sqrt{3}+1),$$

and use the upper sign, we obtain

$$x=-\tfrac{1}{2}\sqrt{(-1)}(\sqrt{3}+1)\pm\sqrt{\{[2\sqrt{3}-2+\sqrt{(-3)}]/[2\sqrt{(-1)}]\}}.$$

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Now we may find (by guess and trial) that $\sqrt{[4\sqrt{(-1)}-4\sqrt{(-3)}-2\sqrt{3}]}$ is $2+\sqrt{(-1)}-\sqrt{(-3)}$; whence

$$x = \tfrac{1}{2}\{\sqrt{(-1)}(\sqrt{3}+1)\pm[2+\sqrt{(-1)}-\sqrt{(-3)}]\} = 1-\sqrt{(-3)}$$

or $-1-\sqrt{(-1)}$, but no method is generally available to extract such square roots.

Sur l'aire d'un polygone régulier étoilé.

Par le Professeur L. CATALÁ.

Nous nous proposons de trouver l'aire $A_{g,e}$ d'un polygone régulier étoilé de genre g et d'espèce e. Soient AD, BC deux côtés qui se coupent en M, l'arc AB étant égal à la g-ième partie de la circonférence. Les triangles semblables et isoscèles CMD, AMB donnent $CD/AB = CM/MB$,

d'où $$(CD+AB)/AB = (CM+MB)/MB = BC/MB,$$

de là $$MB = (AB.BC)/(AB+CD);$$

par suite : périmètre apparent $= (2g.AB.BC)/(AB+CD)$, puis, en désignant par a l'apothème, $A_{g,e} = (g.AB.BC)/(AB+CD)\,a$. Mais $AB = \lambda_{g,1}$, $BC = \lambda_{g,e}$, CD = corde de l'arc $(2e-1)/g = c$; donc

$$A_{g,e} = (g.\lambda_{g,1}.\lambda_{g,e})/(\lambda_{g,1}+c)\,a.$$

Applications.

I. *Pentagone régulier de 2-ième espèce.*—On a

$$\lambda_{5,1} = \tfrac{1}{2}R\sqrt{(10-2\sqrt{5})},\quad \lambda_{5,2} = \tfrac{1}{2}R\sqrt{(10+2\sqrt{5})},\quad c = \tfrac{1}{2}R\sqrt{(10+2\sqrt{5})},$$
$$a = \tfrac{1}{4}R(\sqrt{5}-1).$$

Opérations faites,

$$A_{5,2} = \tfrac{5}{8}(\sqrt{5}-1)\,[\sqrt{(10+2\sqrt{5})}-\sqrt{(10-2\sqrt{5})}]\,R^2.$$

II. *Décagone régulier de 3-ième espèce.*—Dans ce cas

$\lambda_{10,1} = \frac{1}{2}R(\sqrt{5}-1)$, $\lambda_{10,3} = \frac{1}{2}R(\sqrt{5}+1)$, $c = 2R$, $a = \frac{1}{4}R\sqrt{(10-2\sqrt{5})}$.

Calculs effectués $A_{10,3} = \frac{5}{2}\sqrt{(50-22\sqrt{5})}\,R^2$.

On Perfect Numbers: some Corrections.

By Lt.-Col. ALLAN CUNNINGHAM, R.E.

In Vol. XLVIII. (1888), pp. 183–191, of the *Reprint* is an interesting "Note on Perfect Numbers," by Mr. R. W. D. CHRISTIE; on p. XXXVI. of the same volume (at foot) there is a *corrigendum* on the above to the effect that "the 9th and 10th perfect numbers appear to be $2^{40}(2^{41}-1)$ and $2^{46}(2^{47}-1)$."

Again in Vol. XLIX., p. 85, of the same work there is a Question, No. 9468, by the same writer: "Show that the 10th perfect number is $2^{40}(2^{41}-1)$," with an apparent "solution" by Prof. I. BRYENS, quoting

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the authority of M. J. Carvallo's *Théorie des Nombres parfaits*, Paris, 1883; the list of perfect numbers on p. 30 of this work gives $2^{28}.(2^{29}-1)$ and $2^{40}.(2^{41}-1)$ as "perfect."

None of the above three instances are, however, perfect numbers, since

$$2^{29}-1 = 233.1103.2089; \quad 2^{41}-1 = 13367.164511353;$$
$$2^{47}-1 = 2351.4513.13264529.$$

The factors of $(2^{29}-1)$ were published by Euler (1772); those of $(2^{41}-1)$ by Plana (1859); those of $(2^{47}-1)$ by Landry (1867).

It was shown by Euler (1772) that all even perfect numbers (P) are contained in the formula $P = 2^{q-1}.M_q$, where $M_q = 2^q-1 =$ prime (which involves $q =$ prime). Thus the finding of even perfect numbers resolves itself into finding prime values of M_q. These numbers M_q have been called "Mersenne's Numbers"; a very good *résumé* of what is known about them is contained in Prof. W. W. Rouse Ball's paper "On Mersenne's Numbers."

The following is believed to be a complete list of the values of q (a prime $\gtrless 257$) which yield prime or composite values of $M_q = 2^q-1$, so far as yet known; this list is compiled from Prof. W. W. Rouse Ball's paper above quoted, with three additions, viz., when $q = 67, 127, 197$ (see footnotes †, ‡, §).

M_q	*Value of q (a prime).*
prime:	2, 3, 5, 7, 13, 17, 19, 31, 61,* 127‡;
composite:	11, 23, 29, 37, 41, 43, 47, 59, 67,† 73, 79, 83, 97, 113, 151, 179, 191, 197,§ 211, 223, 233, 239, 251;
undetermined:	71, 89, 101, 103, 107, 109, 137, 139, 149, 157, 163, 167, 173, 181, 193, 199, 227, 229, 241, 257.

The chance of discovery of any more Mersenne's primes is seen to be small, as the numbers run so high; M_{71} being the lowest undetermined case.

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* M_{61} was shown prime by Herr Seelhoff; see *Zeitschrift für Mathem. und Phys.*, v. xxi., 1886, p. 178; the reasoning is not quite convincing, so that verification is desirable.

† M_{67} is considered composite by Ed. Lucas; see *Sur la Théorie des Nombres premiers*, Turin, 1876, p. 11. This result appears to depend on an indirect process, *i.e.*, by the failure of a certain test of primality, involving a long arithmetical search; any mistake in this would vitiate the result, so that verification is desirable.

‡ M_{127} appears to have been considered prime by Mersenne (but on what grounds is unknown); its primality has been verified by Ed. Lucas; see *Recherches sur plusieurs ouvrages de* Léonard de Pise, Rome, 1877, p. 26. M. Lucas's method of testing primality has the advantage of being a direct process; the work was only done once, so that verification is desirable.

§ $M_{197} \equiv 0 \pmod{7487}$; this divisor was discovered by the present writer.

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As to odd perfect numbers, Mr. CARVALLO's paper, above quoted, contains an attempt (pp. 1–17) to prove the non-existence of odd perfect numbers; but the reasoning is defective; in fact the crucial step (pp. 9, 10) contains a *petitio principii*.

15384. (R. TUCKER, M.A.)—ABC is a triangle. On AB, AC are described segments of circles containing angles equal to 2B, 2C respectively, and on BC a segment BOC, O the circum-centre. These circles intersect in P, the co-ordinates of which are

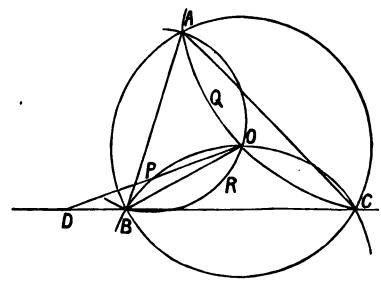
$$\frac{\alpha}{2\cos A} = \frac{\sin(2C-B)}{\sin 2C}\beta = \frac{\sin(2B-C)}{\sin 2B}\gamma.$$

Points Q, R are got in a similar manner from the other sides. Does the circle round PQR pass through O?

Solution by C. E. YOUNGMAN, M.A.

Invert the figure with respect to the circle ABC, and let D, E, F be the inverses of P, Q, R; then, since the circle BPOC becomes the line BC, D is where OP cuts BC. And the triangles OPA, OAD are similar, so that

$$\angle OAD = \angle OPA = \angle CPA - \angle CPO = 2C - \angle CBO;$$



therefore $\angle BAD = 2C - B$.

Hence $BD : DC = -c\sin(2C-B) : b\sin(2B-C)$; and we see that D, E, F will be collinear (and O, P, Q, R concyclic) only if

$$\Pi\sin(2B-C) = \Pi\sin(2C-B).$$

This is equivalent to $\Sigma\cos 2A \times \Pi\sin(B-C) = 0$; so that, apart from isosceles triangles, we must have $\Sigma\cos 2A = 0$, or (the same thing) $\Sigma\cos A\cos(B-C) = 0$. But, if so, then

$$\sin(2B-C) : \sin(2C-B) = \cos(A-B) : \cos(C-A),$$

and so on; therefore the trilinear equation of DEF will be $\Sigma\alpha\cos(B-C) = 0$. This shows that DEF passes through O, and therefore through P, Q, R also; and, after all, O, P, Q, R are not concyclic.

Various interesting results follow from the condition $\Sigma\cos 2A = 0$. Thus (1) the triangle is obtuse-angled, say at A.

(2) $\cos A\cos B\cos C = -\frac{1}{4}$. (3) $\Sigma\cot A.\Sigma\sin 2A = 3$.

(4) $\Sigma(\cot A + \cot 3A) = 0$. (5) $\Sigma a^3\cos A = 0$.

(6) $\sin 3A : \sin 3B : \sin 3C = \cos A : \cos B : \cos C$.

(7) $a^2 + b^2 + c^2 = 6R^2$. (8) $OH^2 = 3R^2$ (H the orthocentre).

(9) The centroid G inverts into H; for $OG.OH = \frac{1}{3}OH^2 = R^2$.

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(10) The pedal triangle of G is similar to that of H, and has angles $2A-\pi$, 2B, 2C. (11) The angles subtended at G by the sides of the triangle are $3A-\pi$, 3B, 3C. (12) At the symmedian point K these angles are $2\pi-2A$, $\pi-2B$, $\pi-2C$. (13) OK is a diameter of the rectangular hyperbola ABCOHK. (14) The BROCARD centre (mid-point of OK) lies on the nine-point circle. (15) The radical axis of the circum- and nine-point circles goes through the nine-point centre. (16) The polar circle is equal to the circum-circle, and goes through K. (17) Mr. TUCKER's "sine-triple-angle circle" is also equal to the circum-circle. (18) The angle A lies between 120° and $\cos^{-1}\frac{1}{8}(1-\sqrt{3})$: which is about $111\frac{1}{2}°$. (19) The NEUBERG point

$$[\alpha \sin(B-C) = \beta \sin(C-A) = \gamma \sin(A-B)]$$

is the orthocentre of the first BROCARD triangle. (20) The director-circle of the minimum ellipse round ABC goes through H. (21) Draw ALL', BMM', CNN' equilateral triangles of the same sign as ABC with their bases in BC, CA, AB. Then LMN, L'M'N' are straight lines intersecting at right angles at K (see *Mathesis*, 1898, p. 134).

15322. (Professor NEUBERG.)—Trouver les cubes parfaits de la forme $7p+1$, p étant un nombre premier.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

Let $7p+1=x^3$; then $7p=x^3-1$; therefore $7p=(x-1)(x^2+x+1)$; and, since 7 and p are both prime, there are only three solutions, given by $7=x^2+x+1$, giving $x=2$, $p=1$, $x^3=8$ or $x=-3$, $p=-4$, $x^3=-27$. Also $7=x-1$, giving $x=8$, $p=73$, $x^3=512$.

15423. (*Revised.*) (S. C. BAGCHI, B.A.)—A rod DE, of variable length, meets two others AB, BC, including a right angle at B. The triangle DBE so formed is held with its plane inclined (in a direction which shall be *definite*) to that of a screen illuminated by a parallel pencil of direct incidence. If DE be moved so as to be always at the same distance from B, prove that the locus of the foot of one of the symmedians of the shadow of BDE is an elliptic arc.

Solution by the PROPOSER.

Since the moving rod is always at a constant distance from B, it envelops a circle of which B is the centre. In the shadow the circular arc will be projected as an elliptic arc of which the two conjugate dia-

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meters are the shadows of the rods including B. The shadow of the moving rod envelops that elliptic arc. Taking the conjugate diameters as axes, we find the equation of the symmedian which passes through the projection of B is $a^2xy' - b^2yx' = 0$, where a, b are the conjugate semi-diameters of the ellipse of projection and (x', y') is the point where the shadow of the moving rod touches the elliptic arc. The equation of the shadow of the moving rod is $xx'/a^2 + yy'/b^2 = 1$. Let (h, k) be a point on the locus in question,

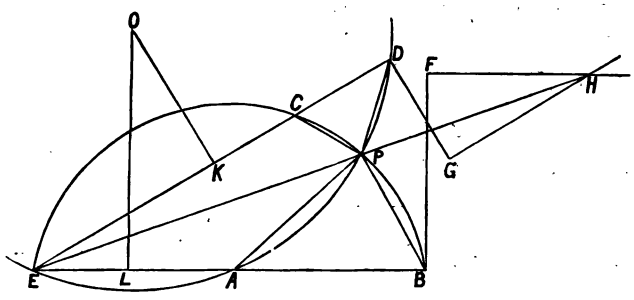
$$hx'/a^2 + ky'/b^2 = 1,\ ha^2y' - kb^2x' = 0,\ x'^2/a^2 + y'^2/b^2 = 1 \ldots \text{(i., ii., iii.)}.$$

Eliminating x', y', we get for the locus $a^2x^2 + b^2y^2 = (x^2 + y^2)^2$. If $a = b$, the locus is the part of ellipse of projection, *i.e.*, the triangle must be so held that either the internal or external bisector of B is parallel to the plane of the screen.

15083. (John Prescott, B.A.)—Given two lines AB and CD in the same plane, but not parallel, find a point P such that the triangles formed by joining P to the extremities of AB and CD shall be similar, the two given lines subtending equal angles at P.

Solutions (I.) *by* D. Biddle; (II.) *by* J. H. Taylor, M.A.; (III.) *by* A. M. Nesbitt, M.A.

(I.) Produce the two lines to meet in E. Draw BF, DG equal to, and at right and angles to, AB, CD respectively, and FH, GH parallel to AB, CD respectively, to meet in H. Join EH. Then EH is clearly the



locus of P such that the perpendiculars from P upon AB, CD are proportionate to AB, CD respectively. Next bisect ED, EA in K, L respectively and draw the perpendiculars KO, LO. Then O is the centre of the circle passing through D, A, E, and this circle cuts EH in P, making $\angle PDC = \angle PAB$.

The circle through B, C, E intersects the circle through D, A, E in P, in the case represented in the diagram, and this suggests an alternative method of solving the question.

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(II.) Let AB, CD be the given lines. Let AC, BD intersect in E (Fig. 1). Draw circles ABE, CED, intersecting in O. Join OA, OB, OC, OE, OD.

$\angle AOB = \angle AEB$
$= \text{suppt. of } \angle CED$
$= \angle COD,$
$\angle ABO = \angle AEO$
$= \angle CEO = \angle CDO;$

therefore the triangles OAB, OCD are equiangular.

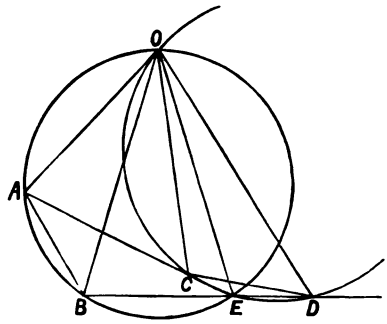


Fig. 1.

Again, let BC, AD intersect in E′ (Fig. 2). Draw circles ABE′, CE′D intersecting in O′. Join O′A, O′B, O′C, O′E′, O′D.

$\angle AO'B = \angle AE'B$
$= \text{suppt. of } \angle CE'D$
$= \angle CO'D,$
$\angle ABO' = \text{suppt. of } \angle AE'O'$
$= \angle O'E'D$
$= \angle O'CD;$

therefore the triangles O′AB, O′CD are equiangular.

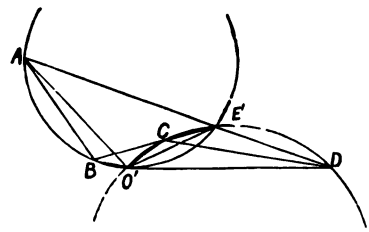
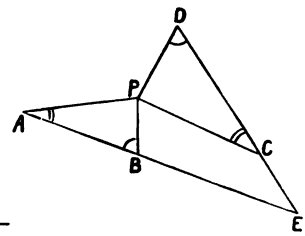


Fig. 2.

(III.) Let AB and CD be produced to meet in E; and let the circles EBD and ECA cut in P; then we have at once

$$\angle PDC = \angle PBA$$

and $$\angle PCD = \angle PAB.$$

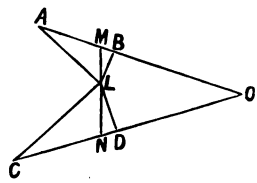


Note by the Proposer.

There is another case not taken into account in the above solutions. It is the case in which a rotation of one of the triangles in the plane would not make it similarly situated with the other triangle.

Let $OB = b$, $OD = d$, $AB = a$, $CD = c$. Then, if we find a point M such that

$$MB(c-a) = a(b-d) \ldots\ldots (1),$$



and mark off ON along OC, equal to OM, we have

$$ND = (b + MB) - d = [a\,MB + a(b-d)]/a = MB \cdot c/a,$$

i.e.,

$$ND/MB = CD/AB.$$

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Now take P in MN so that MP/PN = AB/CD; then, since ∠BMP = ∠DNP, therefore the two triangles BML, DNL are similar. In the same way the two triangles AMP, CNP can be proved similar. Thus P is one of the points required.

15363. ("UNSIGNED.")—$\Sigma = 0$, $\Sigma' = 0$ are the tangential equations of two conics of equal area Δ; α is the envelope of a chord of constant length drawn in Σ; α' is the envelope of an equal chord drawn in Σ'. Show that the envelope of the common tangents of α and α' is $k\Sigma\Sigma' = \pi^4/\Delta^4$, and the conic $\Sigma = \Sigma'$, where k is a constant, inserted to keep the dimensions correct; also when the areas are not equal the envelope is a curve of the sixth order.

Solution by Professor NANSON.

At the meets of the line P and the conic S where $P = (lmn)(xy1)$, $S = (abcfgh)(xy1)^2$ we have, by elimination of y, $c_1x^2 - 2g_1x + a_1 = 0$ where a_1, c_1, g_1 are the co-factors of a, c, g in the bordered discriminant Σ of S, P. Hence, if (x_1, y_1), (x_2, y_2) are the ends of the chord S, P,

$$(x_1 - x_2)^2 = 4(g_1^2 - c_1a_1)c_1^{-2} = 4\Sigma m^2 c_1^{-2}.$$

Hence the square of half the chord S, P is $\Sigma(l^2 + m^2)c_1^{-2}$. Thus, if P make equal chords in S, S', $\Sigma c'_1{}^2 = \Sigma' c_1^2$; so that the envelope of P is a class sextic.

If S, S' are circles, the sextic reduces to the focoids and the parabola $C\Sigma' = C'\Sigma$ where C, C' are the co-factors of c, c' in the discriminants of S, S'.

If S, S' have the same area, then $c^2C' = c'^2C = \lambda c^2c'^2$, say, and if S, S' have *the same centre*, so that we can take f, g, f', g' to be zero, then

$$\Sigma = cc_1 - Cn^2 = c(c_1 - c\lambda n^2);$$

so that the envelope is $cc'_1{}^2(c_1 - c\lambda n^2) = c'c_1^2(c_1' - c'\lambda n^2)$,

and this breaks up into $cc'_1 = c'c_1$, $c_1c'_1 = \lambda n^2(cc'_1 + c'c_1)$, that is, into the parabola $C\Sigma' = C'\Sigma$ and the parabolic class quartic $\Sigma\Sigma' = CC'n^4$.

Additional Note on Question 15282.

[*V. Educational Times* for November, 1903, and *Reprint*, Vol. v. (New Series), p. 89.]

By G. H. HARDY, M.A.

I agree with θ. β. that the value of the integral was in all probability known to CAUCHY. The result of the question and the corresponding

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formula for $\int_0^\infty \frac{x\cos ax}{1+x^2}\,dx$ correspond exactly to the two formulæ

$$\int_0^\infty \frac{\cos ax}{1+x}\,dx = \int_0^\infty \frac{xe^{-ax}}{1+x^2}\,dx, \quad \int_0^\infty \frac{\sin ax}{1+x}\,dx = \int_0^\infty \frac{e^{-ax}}{1+x^2}\,dx, \quad (a>0)$$

which express the integrals on the right in terms of the functions $\int \frac{\sin x}{x}\,dx$, $\int \frac{\cos x}{x}\,dx$, and these formulæ are given in CAUCHY's "Mémoire sur les Intégrales définies" (*Œuvres*, t. I., p. 377).

The question is such an interesting one that I may perhaps be allowed to add a few remarks on some other ways in which it can be solved. In the first place, SCHLÖMILCH's solution is far from "elementary." He proves first that

$$\int_0^\infty \{e^{-t}\,\mathrm{li}\,(e^t) - e^t\,\mathrm{li}\,(e^{-t})\}\sin ut\,dt = \pi/(1+u^2),$$

and deduces the result by means of FOURIER's double-integral theorem (*Compendium*, Vol. II., pp. 201–3). I gave another solution recently in a paper "The Theory of CAUCHY's Principal Values" (*Proc. Lond. Math. Soc.*, Vol. XXXV., p. 96). Since $\sin ay = -\frac{2a}{\pi}\,\mathrm{P}\int_0^\infty \frac{\cos xy}{x^2-a^2}\,dx$,

$$\int_0^\infty \frac{\sin ay}{1+y^2}\,dy = -\frac{2a}{\pi}\int_0^\infty \frac{dy}{1+y^2}\,\mathrm{P}\int_0^\infty \frac{\cos xy}{x^2-a^2}\,dx = -\frac{2a}{\pi}\,\mathrm{P}\int_0^\infty \frac{dx}{x^2-a^2}\int_0^\infty \frac{\cos xy}{1+y^2}\,dy$$

$$= -a\mathrm{P}\int_0^\infty \frac{e^{-x}\,dx}{x^2-a^2} = \tfrac{1}{2}\{e^{-a}\,\mathrm{li}\,(e^a) - e^a\,\mathrm{li}\,(e^{-a})\}.$$

This argument of course needs exceedingly careful theoretical justification.

Other solutions may be deduced from the theory of divergent series and integrals. Thus, if the integral of the question be denoted by u,

$$\frac{du}{da} = \int_0^\infty \frac{y\cos ay}{1+y^2}\,dy, \qquad \frac{d^2u}{da^2} = -\mathrm{G}\int_0^\infty \frac{y^2\sin ay}{1+y^2}\,dy,$$

$$\frac{d^2u}{da^2} - u = -\mathrm{G}\int_0^\infty \sin ay\,dy = -\frac{1}{a},$$

which is the differential equation found otherwise by Mr. DICK. The meaning of the symbol G is explained in a paper in the last issue of the *Quarterly Journal*.

Or again expand $1/(1+y^2)$ in the series $1-y^2+y^4-\ldots$, divergent if $y>1$. Take the divergent integral term by term. Since

$$\mathrm{G}\int_0^\infty \sin ay\,dy = \frac{1}{a}, \qquad \mathrm{G}\int_0^\infty y^{2n}\sin ay\,dy = \frac{2n!}{a^{2n+1}},$$

we obtain $u = 1/a - 2!/a^3 + 4!/a^5 - \ldots$, a divergent series which gives the result of the question on summation by M. BOREL's method.

The last two methods are of course really symbolic, the argument being really a shorthand representation of another which is much more involved.

Finally, I would add that Mr. DICK's definition of the function $\mathrm{li}\,e^a$, though slightly inaccurate in form, as at first stated, is really very interesting, as (if modified as suggested in my solution) it gives a formula applicable to all positive or negative values of a and involving only *unconditionally* convergent integrals.

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Geometrical Note on FRÉGIER'S *Theorem.*

By R. F. DAVIS, M.A.

Let OPQ be a triangle having the angle at O a right angle and OR perpendicular upon PQ. Then, if any variable transversal meet OP, OQ, OR respectively in p, q, r, it may be shown that

$$1/(\mathrm{OP}\cdot \mathrm{O}p) \pm 1/(\mathrm{OQ}\cdot \mathrm{O}q) = 1/(\mathrm{OR}\cdot \mathrm{O}r).$$

Now Dr. C. TAYLOR, in the new edition of his *Elementary Geometry of Conics*, proves that, if OP be any chord of a conic which meets in p the focal parallel to the tangent at O, then $\mathrm{OP}\cdot \mathrm{O}p = f\cdot \mathrm{SO}$, where f is the focal chord parallel to OP (Ex. 107, p. 63). Similarly, $\mathrm{OQ}\cdot \mathrm{O}q = g\cdot \mathrm{SO}$, where g is the focal chord parallel to OQ and Spq is parallel to the tangent at O. Now $(1/f \pm 1/g)/\mathrm{SO}$ is constant when OP, OQ (and consequently f, g) are at right angles. Thus $\mathrm{OR}\cdot \mathrm{O}r$, and therefore O$r$, is constant.

10143. (Professor DÉPREZ.)—Sur les côtés BC, CA, AB du triangle sphérique ABC on prend, dans le même sens, trois arcs égaux, BA′ = CB′ = AC′. Démontrer que le point de rencontre des médianes du triangle A′B′C′ décrit un grand cercle.

Solution by A. M. NESBITT, M.A.

Let O be the centre of the sphere, and take rectangular planes through OA, OB, OC as those of yz, zx, xy respectively. Let the direction cosines of OA, OB, OC be (o, m_1, n_1), (l_2, o, n_2), (l_3, m_3, o) and those of OA′, OB′, OC′ be $(\lambda_1, \mu_1, \nu_1)$, $(\lambda_2, \mu_2, \nu_2)$, $(\lambda_3, \mu_3, \nu_3)$. Then, if G be the intersection of the medians, we shall have plainly $x/\Sigma\lambda = y/\Sigma\mu = z/\Sigma\nu$ as the equations to OG. Now, since OA′ lies in the plane OBC, $\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ l_2 & o & n_2 \\ l_3 & m_3 & o \end{vmatrix} = 0$, which gives us the right to assume $\lambda_1 = q_1 l_2 + r_1 l_3$, $\mu_1 = r_1 m_3$, $\nu_1 = q_1 n_2$. Similarly we may put $\lambda_2 = q_2 l_3$, $\mu_2 = q_2 m_3 + r_2 m_1$, $\nu_2 = r_2 n_1$, and $\lambda_3 = r_3 l_2$, $\mu_3 = q_3 m_1$, $\nu_3 = q_3 n_1 + r_3 n_2$.

These yield at once

$$q_1^2 + r_1^2 + 2q_1 r_1 l_2 l_3 = q_2^2 + r_2^2 + 2q_2 r_2 m_3 m_1 = q_3^2 + r_3^2 + 2q_3 r_3 n_1 n_2 = 1.$$

But, since the arcs BA′, CB′, AC′ are equal (and each equal to θ, say), we have $\lambda_1 l_2 + \nu_1 n_2 = \lambda_2 l_3 + \mu_2 m_3 = \mu_3 m_1 + \nu_3 n_1$, or, on substituting for λ, μ, ν,

$$q_1 + r_1 l_2 l_3 = q_2 + r_2 m_3 m_1 = q_3 + r_3 n_1 n_2 = \cos\theta.$$

Squaring these and subtracting from the above, we find $r_1^2 = r_2^2 = r_3^2$. Thus the equations to OG become

$$x/\left[q_1 l_2 + q_2 l_3 + r_1 (l_2 \pm l_3)\right] = y/\left[q_2 m_3 + q_3 m_1 + r_1 (m_3 \pm m_1)\right] = z/\left[q_3 n_1 + q_1 n_2 \pm r_1 n_1 \pm r_1 n_2\right].$$

On substituting for q_1, q_2, q_3 in terms of r_1 and $\cos\theta$, OG is represented by equations such as $x/(\mathrm{A}r_1 + \mathrm{B}\cos\theta) = y/(\mathrm{C}r_1 + \mathrm{D}\cos\theta) = z/(\mathrm{E}r_1 + \mathrm{F}\cos\theta)$; so that OG lies in the plane $\mathrm{P}x + \mathrm{Q}y + \mathrm{R}z = 0$, where $\mathrm{PA} + \mathrm{QC} + \mathrm{RE} = 0$, $\mathrm{PB} + \mathrm{QD} + \mathrm{RF} = 0$. Hence G describes a great circle.

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15095. (Professor NEUBERG.)—ABC étant un triangle donné, trouver un point M tel que si A′, B′, C′ sont les points de rencontre de BC, CA, AB avec les droites AM, BM, CM, M soit le centre de gravité des points A′, B′, C′ pour les masses données α, β, γ.

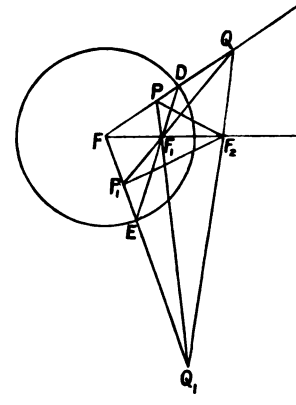
Solution by C. E. YOUNGMAN, M.A.

Through A, B, C draw parallels to BC, CA, AB, forming the triangle $A_0B_0C_0$. The point M is always the same mass-centre in $A_0B_0C_0$ as it is in A′B′C′. For suppose M is the centre of ABC for masses λ, μ, ν; then $\frac{1}{2}\mu$ at B and $\frac{1}{2}\nu$ at C are equivalent to $\frac{1}{2}(\mu+\nu)$ at A′, and so on; so that the system is equivalent to $\frac{1}{2}(\mu+\nu)$, $\frac{1}{2}(\nu+\lambda)$, $\frac{1}{2}(\lambda+\mu)$ at A′, B′, C′. And, on the other hand, we might replace λ at A by $\frac{1}{2}\lambda$ at B_0 and $\frac{1}{2}\lambda$ at C_0, and so on; thus getting the system $\frac{1}{2}(\mu+\nu)$, $\frac{1}{2}(\nu+\lambda)$, $\frac{1}{2}(\lambda+\mu)$ at A_0, B_0, C_0. Hence to solve the problem we have only to divide A_0B_0 at E in the ratio $\beta:\alpha$, and A_0C_0 at F in the ratio $\gamma:\alpha$, and join B_0E, C_0F.

4487. (B. WILLIAMSON, M.A.)—Suppose F, F_1, F_2 to be the three foci of a Cartesian oval, F_2 being the exterior focus, and let $FF_1 = c_2$, $FF_2 = c_1$, $F_1F_2 = c$; then, if $mr + lr_1 = nc_2$ be the equation of the inner oval referred to F and F_1, show that its equation referred to F and F_2 is $nr + lr_2 = mc_1$; and referred to F_1 and F_2 is $mr_2 - nr_1 = lc$. Also prove that the corresponding equations of the outer oval are obtained by changing l into $-l$ in the preceding.

Solution by H. J. WOODALL, A.R.C.S.

With F as centre and radius nc_2/m describe a circle; through F_1 draw a chord DE to cut the circle in D and E; join FD, FE and produce as shown in the figure. Let FD cut the inner oval at P, the outer oval at Q.



Then $PD = FD - FP = (nc_2/m - r)$
$= lr_1/m = lPF_1/m$;
Also $DQ = FQ - FD = (r' - nc_2/m)$
$= lr'_1/m = lQF_1/m$;
therefore $PD : DQ = PF_1 : QF_1$;
therefore $\angle PF_1D = \angle QF_1D = \angle P_1F_1E = \angle Q_1F_1E$, where P_1, Q_1 are the points in which FE is cut by QF_1, PF_1 respectively, and F_1, Q_1 are respectively on the inner and outer ovals. Also FDE is an isosceles triangle; therefore the triangles PF_1D, P_1F_1E are equiangular. Similarly, the triangles F_1EQ_1, F_1DQ are equiangular. It follows therefore that Q_1, P_1, P, Q are concyclic; and, similarly, Q_1, E, D, Q are concyclic also. Now draw QF_2, making $\angle F_1QF_2 = \angle F_1FP_1$; then the triangles F_1QF_2, F_1FP_1 will be

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equiangular also. Similarly, the triangles FPF_1 and FF_2Q are equiangular.

Now in the triangle FPF_1 we have $m\,FP + l\,F_1P = n\,FF_1$; therefore in the triangle FF_2Q we have $m\,FF_2 + l\,F_2Q = n\,FQ$ *i.e.*, $mc_1 + lr'_2 = nr'$. Now divide PQ in D′ so that $PD' : PF_2 = D'Q : QF_2 = l : n$; therefore $r' = FP + PD' + D'Q = r + l(r_2 + r'_2)/n$; therefore $mc_1 = nr' - lr'_2 = nr + lr_2$.

Hence the equation to the inner oval referred to F and F_2 is $nr + lr_2 = mc_1$. At the same time we show that the change from the inner to the outer oval is accompanied by change of sign of l.

Again, take the triangle F_1FP_1; $m\,FP_1 + l\,F_1P_1 = n\,FF_1$; therefore in the similar triangle F_1QF_2 we have $m\,QF_2 + l\,F_1F_2 = n\,F_1Q$; therefore $mr'_2 + lc = nr'_1$; or, changing to the inner oval, $mr_2 - nr_1 = lc$. (See WILLIAMSON's *Differential Calculus*, p. 375.)

15406. (Professor NANSON.)—Show that the tangent of the angle between the axes of two parabolas is the harmonic mean of the tangents of the angles made by the tangents to either curve at the four common points with the axis of that curve.

Solutions (I.) *by* R. F. DAVIS, M.A.; (II.) *by* Rev. T. WIGGINS, B.A.; (III.) *by* Professor BROWN.

(I.) Denoting the point on the parabola $y^2 = 4ax$ whose co-ordinates are n^2a, $2na$ by the parameter n, the general equation to a conic intersecting the parabola in the points n_1, n_2, n_3, n_4 is

$$\lambda(y^2 - 4ax) + \{(n_1 + n_2)y - 2x - 2n_1n_2a\}\{(n_3 + n_4)y - 2x - 2n_3n_4a\} = 0.$$

This will, by properly choosing λ, represent another parabola having $y\,\Sigma n_1 - 4x = 0$ as a diameter. Hence the tangent of the angle between the axes $= 4/\Sigma n_1 = 4/\Sigma \cot\theta_1$, where θ_1 is the angle the tangent at n_1 makes with the axis of x.

(II.) Take the parabolas

$$y^2 = 4ax \quad \text{and} \quad (x\cos\alpha + y\sin\alpha - p)^2 = 2\kappa(x\sin\alpha - y\cos\alpha - p').$$

If $(am^2, 2am)$ is a common point, $-1/m_1$, $-1/m_2$, $-1/m_3$, $-1/m_4$ are the tangents of the angles made with the axis by the tangents to $y^2 = 4ax$ at the common points, where m_1, m_2, m_3, m_4 are the roots of

$$(am^2\cos\alpha + 2am\sin\alpha - p)^2 = 2\kappa(am^2\sin\alpha - 2am\cos\alpha - p'),$$

$m_1 + m_2 + m_3 + m_4 = -4\tan\alpha$; therefore

$$\Sigma\, 1/(-1/m_1) = 4/(-\cot\alpha) \quad \text{and} \quad -\cot\alpha = \tan(\angle \text{ between axes}).$$

(III.) Let the equations of the parabolas be

$$y^2 = 4ax \quad \text{and} \quad (px + qy)^2 + rx + sy + t = 0.$$

$y = mx + a/m$ is a tangent to the first at $(a/m^2, 2a/m)$; if this point lies on the other, $(pa/m^2 + 2qa/m)^2 + r(a/m^2) + 2sa/m + t = 0$. The roots of this biquadratic in m are the tangents of the angles which are made with the

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axis of $y^2-4ax=0$ by the four tangents drawn to $y^2-4ax=0$ at its four points of intersection with $(px+qy)^2+rx+sy+t=0$. Denote these roots by t_1, t_2, t_3, t_4, and by t_a the tangent of the angle which the axis of the second parabola (parallel to $px+qy=0$) makes with the axis of the first (axis of x).

$$1/t_1+1/t_2+1/t_3+1/t_4=-4pqa^2/p^3a^3=-4q/p=+4/t_a.$$

15361. (G. H. HARDY, M.A.)—The area of the diagonal triangle of the quadrangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) (in areal co-ordinates) is numerically $\dfrac{54\Delta\, D_{234}\, D_{314}\, D_{124}\, D_{123}}{D_{23,14}\, D_{31,24}\, D_{12,34}}$,

where $D_{\lambda\mu\nu}=\begin{vmatrix} x_\lambda & y_\lambda & z_\lambda \\ x_\mu & y_\mu & z_\mu \\ x_\nu & y_\nu & z_\nu \end{vmatrix}$, $\quad D_{\alpha\beta,\gamma\delta}=\begin{vmatrix} 1 & x_\alpha-x_\beta & x_\gamma-x_\delta \\ 1 & y_\alpha-y_\beta & y_\gamma-y_\delta \\ 1 & z_\alpha-z_\beta & z_\gamma-z_\delta \end{vmatrix}$,

and Δ is the area of the triangle of reference.

Solution by the PROPOSER.

If we denote (x_1, y_1, z_1), (x_2, y_2, z_2), ..., by (1), (2), ..., the co-ordinates of the lines (12), (34) are

$y_1z_2-y_2z_1 : z_1x_2-z_2x_1 : x_1y_2-x_2y_1$ and $y_3z_4-y_4z_3 : z_3x_4-z_4x_3 : x_3y_4-x_4y_3$;

and their point of intersection is

$$(z_1x_2-z_2x_1)(x_3y_4-y_4x_3)-(x_1y_2-x_2y_1)(z_3x_4-z_4x_3) : \ldots : \ldots.$$

Hence the area of the diagonal triangle is $\dfrac{Kf_{12}}{D'_{23,14}\, D'_{31,24}\, D'_{12,34}}$, where K is a constant, f_{12} a homogeneous function of degree 12, and

$$D'_{\alpha\beta,\gamma\delta}=\begin{vmatrix} 1 & y_\alpha z_\beta-y_\beta z_\alpha & y_\gamma z_\delta-y_\delta z_\gamma \\ 1 & z_\alpha x_\beta-z_\beta x_\alpha & z_\gamma x_\delta-z_\delta x_\gamma \\ 1 & x_\alpha y_\beta-x_\beta y_\alpha & x_\gamma y_\delta-x_\delta y_\gamma \end{vmatrix}.$$

It is easy to prove (using the equations $x_1+y_1+z_1=1$, ...) that $D'_{\alpha\beta,\gamma\delta}=\frac{1}{8}D_{\alpha\beta,\gamma\delta}$. Moreover, it is evident that the diagonal triangle degenerates into a straight line when three of the vertices of the quadrangle are collinear: and therefore f_{12} is a constant multiple of $D_{234}\, D_{314}\, D_{124}\, D_{123}$. We may suppose the constant multiplier absorbed in K. To determine K we may suppose either (i.) that the quadrangle is formed by the vertices of the triangle of reference and the orthocentre, in which case the diagonal triangle is the pedal triangle; or (ii.) that the quadrangle is formed by the four in-centres, in which case the diagonal triangle is the triangle of reference. In either case we find $K=-2\Delta$, and the result follows.

15308. (A. M. NESBITT, M.A.)—The two tangents (other than the y-axis) which can be drawn from the point $(0, b)$ to the curve $y^3+x^3=ax^2$ have for equation $4ax^2=27b^3(x/a+y/b-1)^2$.

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Solutions (I.) *by* FRANCES E. CAVE; (II.) *by* C. M. ROSS.

(I.) The equation of the tangent at (x_1, y_1) is

$$x(3x_1^2-2ax_1)+3yy_1^2-ax_1^2=0.$$

If this passes through $(0, b)$, we have $ax_1^2 = 3by_1^2$, and hence, from the equation of the curve, $(y_1-b)/3b + x_1/a = \frac{2}{3}$; therefore

$$ax_1^2 = \tfrac{27}{4}b^3[x_1/a+(y_1-b)/b]^2.$$

But the equation of the tangent is $(y-b)/x = (y_1-b)/x_1$; therefore the equation of the two tangents is $4ax^2 = 27b^3(x/a+y/b-1)^2$.

(II.) The equation of the tangent to the cubic $y^3+x^3 = ax^2$ is

$$(X-x)\,\partial u/\partial x+(Y-y)\,\partial u/\partial y = 0 \quad \text{.....(i.).}$$

Now $\partial u/\partial x = 3x^2-2ax$ and $\partial u/\partial y = 3y^2$. Hence (i.) becomes

$$X(3x^2-2ax)+3Yy^2-ax^2=0 \quad \text{.....(ii.).}$$

Again, let (h, k) be any point on either of the tangents drawn from $(0, b)$; the equation of the line joining $(0, b)$ to (h, k) is $X(k-b)-Yh+hb=0$. If this line touch the cubic, then $(3x^2-2ax)/(k-b) = 3y^2/-h = ax^2/hb$; therefore $(3hb+ak-ab)x = 2ahb$ and $y = x\sqrt{(a/3b)}$ ..... (iii.). But $x^3+y^3 = ax^2$; therefore $(2h/3b)\sqrt{(a/3b)} = h/a+k/b-1$ [by (iii.)]. Squaring, $4h^2a = 27b^3(h/a+k/b-1)^2$; therefore the pair of tangents required is $4ax^2 = 27b^3(x/a+y/b-1)^2$.

15195. (R. TUCKER, M.A.)—Prove

$$\begin{vmatrix} \cos B\cos C & -p\cos C & p\cos B \\ p'\cos C & \cos C\cos A & -p'\cos A \\ -p''\cos B & p''\cos A & \cos A\cos B \end{vmatrix} = \Pi(\cos A)\Pi(\cos B+\cos C-\cos A),$$

where $p = \cos B-\cos C$, $p' = \cos C-\cos A$, $p'' = \cos A-\cos B$.

Solutions (I.) *by the* PROPOSER; (II.) *by* A. M. NESBITT, M.A.; (III.) *by* F. W. REEVES, B.A.

(I.) This is equivalent to—

$$\begin{vmatrix} \beta\gamma & -p\gamma & p\beta \\ p'\gamma & \gamma\alpha & -p'\alpha \\ -p'' & \beta p''\alpha & \alpha\beta \end{vmatrix} = \alpha\beta\gamma(\alpha\beta\gamma+p'p''\alpha+p''p\beta+pp'\gamma)$$
$$= \alpha\beta\gamma(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)(\alpha+\beta-\gamma) = \ldots.$$

(II.) Denoting $\cos A$, $\cos B$, $\cos C$ by α, β, γ, the determinant becomes

$$\begin{vmatrix} \beta\gamma & \gamma^2-\beta\gamma & \beta^2-\beta\gamma \\ \gamma^2-\gamma\alpha & \gamma\alpha & \alpha^2-\gamma\alpha \\ \beta^2-\beta\alpha & \alpha^2-\beta\alpha & \alpha\beta \end{vmatrix}.$$

Now $\alpha = 0$ makes this vanish. Since the minor of $\beta\gamma$ becomes $\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$, hence α, β, γ are factors. Also the substitution $\alpha = \beta+\gamma$ makes the second and third lines identical; therefore $\Pi(\beta+\gamma-\alpha)$ is a factor, and, since the determinant is of the sixth degree, it is equal to $\alpha\beta\gamma\Pi(\beta+\gamma-\alpha)$ (since its value is clearly α^6 when $\alpha = \beta = \gamma$).

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(III.) $\begin{vmatrix} \cos B\cos C & -p\cos C & p\cos B \\ p'\cos C & \cos C\cos A & -p'\cos A \\ -p''\cos B & p''\cos A & \cos A\cos B \end{vmatrix}$

$= \begin{vmatrix} \cos^2 C & -p\cos C & p^2 \\ \cos^2 C & \cos C\cos A & \cos^2 A \\ p''^2 & p''\cos A & \cos^2 A \end{vmatrix}$ (by keeping middle column unchanged and adding it respectively to the first and third columns for new first and third columns)

$= \begin{vmatrix} 0 & \cos C(\cos A + p) & (\cos^2 A - p^2) \\ \cos^2 C & \cos C\cos A & \cos^2 A \\ \cos^2 C - p''^2 & \cos A(\cos C - p') & 0 \end{vmatrix}$ (by keeping middle row unchanged and subtracting first and third rows from it for new first and third rows respectively)

$= \cos^2 C \begin{vmatrix} \cos A(\cos C - p'') & 0 \\ \cos C(\cos A + p) & (\cos^2 A - p^2) \end{vmatrix}$

$+ (\cos^2 C - p''^2) \begin{vmatrix} \cos C(\cos A + p) & (\cos^2 A - p^2) \\ \cos C\cos A & \cos^2 A \end{vmatrix}$

$= \cos A\cos^2 C\,\Pi(\cos B + \cos C - \cos A) + p\cos A\cos C\,\Pi(\cos B + \cos C - \cos A)$

$= \cos A\cos C(\cos C + p)\,\Pi(\cos B + \cos C - \cos A)$

$= \Pi\cos A\,\Pi(\cos B + \cos C - \cos A).$

15392. (Robt. W. D. Christie.)—The sum of the series

$$1 + \frac{4}{3} + \frac{11}{3^2} + \frac{29}{3^3} + \ldots + \frac{a_n}{3^{n-1}} \text{ ad inf.}$$

is thrice a square integer [$a_n = 3a_{n-1} - a_{n-2}$].

Solutions (I.) *by the* Rev. T. Wiggins, B.A.; (II.) *by* H. J. Woodall, A.R.C.S.; (III.) *by* Professor Joseph Brown, C. M. Ross, *and* Professor Sanjána, M.A.

(I.) Let the n-th term be u_n; therefore, since $a_n = 3a_{n-1} - a_{n-2}$, $u_n = u_{n-1} - \frac{1}{3}u_{n-2}$; therefore $\frac{1}{3}u_0 = u_1 - u_2$, $\frac{1}{3}u_1 = u_2 - u_3$, .., or $\frac{1}{3}s = u_1 = \frac{4}{3}$; therefore $s = 3 \cdot 2^2 = 12$.

(II.) Using the scale of relation, we get $S_n = 12 + 3\left(\frac{a_{n-1}}{3^{n-1}} - \frac{8a_n}{3^n}\right)$.

Now $\mathrm{Lt}_{n=\infty} \frac{a}{3^n} = \mathrm{Lt}_{n=\infty}\left(\frac{a_{n-1}}{3^{n-1}} - \frac{a_{n-2}}{3^n}\right) = \mathrm{Lt}_{n=\infty}\left(\frac{a_n}{3^n}\right) - \frac{1}{3}\mathrm{Lt}\left(\frac{a_n}{3^n}\right)$,

and therefore $= 0$; therefore $S_\infty = 12$.

(III.)
$$S = 1 + \frac{4}{3} + \frac{11}{3^2} + \frac{29}{3^3} + \ldots + \frac{a_n}{3^{n-1}} + \ldots$$
$$-3.\frac{1}{3}.S = -\frac{3}{3} - \frac{12}{3^2} - \frac{33}{3^3} - \ldots - \frac{3a_{n-1}}{3^{n-1}} - \ldots$$
$$\frac{1}{3^2}.S = + \frac{1}{3^2} + \frac{4}{3^3} + \ldots + \frac{a_{n-2}}{3^{n-1}} + \ldots$$

Add: $\frac{1}{3^2}.S = 1 + \frac{1}{3}$;

therefore $S = 3^2\left(1 + \frac{1}{3}\right) = 12 = 3$ times a square integer.

Professor Sanjána remarks:—This is a mere coincidence, the second term being 4. More generally, if $u_n + u_{n+2} = ku_{n+1}$, and the series $u_0 + u_1/k + u_2/k^2 + u_3/k^3 + \ldots$ is convergent, we get the sum to be ku_1.

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